

A COLORED $\mathfrak{sl}(N)$ -HOMOLOGY FOR LINKS IN S^3

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ABSTRACT. Fix an integer $N \geq 2$. To each diagram of a link colored by $1, \dots, N$, we associate a chain complex of graded matrix factorizations. We prove that the homotopy type of this chain complex is invariant under Reidemeister moves. When every component of the link is colored by 1, this chain complex is isomorphic to the chain complex defined by Khovanov and Rozansky in [18]. We call the homology of this chain complex the colored $\mathfrak{sl}(N)$ -homology and prove that it decategorifies to the Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial of links colored by exterior powers of the defining representation.

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1. INTRODUCTION

1.1. Background. In the early 1980s, Jones [14] defined the Jones polynomial, which was generalized to the HOMFLY-PT polynomial in [9, 32]. Later, Reshetikhin and Turaev [35] constructed a large family of polynomial invariants for framed links whose components are colored by finite dimensional representations of a complex semisimple Lie algebra, of which the HOMFLY-PT polynomial is the special example corresponding to the defining representation of $\mathfrak{sl}(N; \mathbb{C})$. In general, the Reshetikhin-Turaev invariants for links are very abstract and hard to evaluate. But, when the Lie algebra is $\mathfrak{sl}(N; \mathbb{C})$ and every component of the link is colored by an exterior power of the defining representation, Murakami, Ohtsuki and Yamada [29] gave an alternative construction of the $\mathfrak{sl}(N)$ -quantum invariant using only elementary combinatorics.

If every component of the link is colored by the defining representation, then [29] gives an alternative definition of the (uncolored) $\mathfrak{sl}(N)$ -HOMFLY-PT polynomial. Based on this, Khovanov and Rozansky [18] categorified the (uncolored) $\mathfrak{sl}(N)$ -HOMFLY-PT polynomial, which generalizes the Khovanov homology [17].

1.2. Main results. In the present paper, we generalize Khovanov and Rozansky's construction in [18] to define a homology for links colored by exterior powers of the defining representation of $\mathfrak{sl}(N; \mathbb{C})$, which categorifies the corresponding Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial.

Unless otherwise specified, N stands for a fixed integer greater than or equal to 2 in the rest of this paper. Also, instead of saying an object is colored by the k -fold exterior power of the defining representation of $\mathfrak{sl}(N; \mathbb{C})$, we will simply say that it is colored by k .

As in [29], given a diagram D of a tangle colored by $1, \dots, N$, we resolve it into a collection of certain special colored and oriented graphs. We call such graphs MOY graphs. To each MOY graph, we associate a graded matrix factorization. Then we construct morphisms between these graded matrix factorizations and build a chain complex $C(D)$ over the homotopy category $\text{hmf}_{R,w}$ of graded matrix factorizations. There are some choices involved in the construction of $C(D)$, including choices of resolutions and markings of D . We will show that the isomorphism type of $C(D)$ does not depend on these choices. Furthermore, we will prove in Section 13 the following theorem, which is the main theorem of this paper.

Theorem 1.1. *Let D be a diagram of a tangle whose components are colored by $1, \dots, N$, and $C(D)$ the chain complex defined in Definition 11.4. Then $C(D)$ is a bounded chain complex over the homotopy category of graded matrix factorizations. $C(D)$ is $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ -graded, where the \mathbb{Z}_2 -grading is the \mathbb{Z}_2 -grading of the underlying matrix factorizations, the first \mathbb{Z} -grading is the quantum grading of the underlying matrix factorizations, and the second \mathbb{Z} -grading is the homological grading. The homotopy type of $C(D)$, with the $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ -grading, is invariant under Reidemeister*

moves. If every component of D is colored by 1, then $C(D)$ is isomorphic to the chain complex defined by Khovanov and Rozansky in [18].

Since the homotopy category $\text{hmf}_{R,w}$ of graded matrix factorizations is not abelian, we can not directly define the homology of $C(D)$. But, as in [18], we can still construct a homology from $C(D)$. Recall that each matrix factorization comes with a differential map d_{mf} . If D is a link diagram, then the base ring R is \mathbb{C} , and $w = 0$. So all the matrix factorizations in $C(D)$ are actually cyclic chain complexes. Taking the homology with respect to d_{mf} , we change $C(D)$ into a chain complex $(H(C(D), d_{mf}), d^*)$ of finite dimensional graded vector spaces, where d^* is the differential map induced by the differential map d of $C(D)$. We define

$$H(D) = H(H(C(D), d_{mf}), d^*).$$

If D is a diagram of a tangle with end points, then R is a graded polynomial ring with homogeneous indeterminants of positive gradings, and w is in the maximal homogeneous ideal \mathfrak{J} of R generated by all the indeterminants. So $(C(D)/\mathfrak{J} \cdot C(D), d_{mf})$ is a cyclic chain complex. Its homology $(H(C(D)/\mathfrak{J} \cdot C(D), d_{mf}), d^*)$ is a chain complex of finite dimensional graded vector spaces, where d^* is the differential map induced by the differential map d of $C(D)$. We define

$$H(D) = H(H(C(D)/\mathfrak{J} \cdot C(D), d_{mf}), d^*).$$

In either case, $H(D)$ inherits the $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ -grading of $C(D)$. We call $H(D)$ the colored $\mathfrak{sl}(N)$ -homology of D . The corollary below follows easily from Theorem 1.1.

Corollary 1.2. *Let D be a diagram of a tangle whose components are colored by $1, \dots, N$. Then $H(D)$ is a finite dimensional $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ -graded vector space over \mathbb{C} . Reidemeister moves of D induce isomorphisms of $H(D)$ preserving the $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ -grading.*

For a tangle T , denote by $H^{\varepsilon,i,j}(T)$ the subspace of $H(T)$ of homogeneous elements of \mathbb{Z}_2 -degree ε , quantum degree i and homological degree j . The Poincaré polynomial $P_T(\tau, q, t)$ of $H(T)$ is defined to be

$$P_T(\tau, q, t) = \sum_{\varepsilon,i,j} \tau^\varepsilon q^i t^j \dim H^{\varepsilon,i,j}(T) \in \mathbb{C}[\tau, q, t]/(\tau^2).$$

Based on the construction by Murakami, Ohtsuki and Yamada [29], we give in Section 14 a re-normalization $\text{RT}_L(q)$ of the Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial for links colored by positive integers (that is, by wedge products of the defining representation of $\mathfrak{sl}(N; \mathbb{C})$.) And we prove there that, for a link L colored by positive integers, the graded Euler characteristic of $H(L)$ is equal to $\text{RT}_L(q)$. More precisely, we have the following theorem.

Theorem 1.3. *Let L be a link colored by positive integers. Then*

$$P_L(1, q, -1) = \text{RT}_L(q).$$

Moreover, define the total color $\text{tc}(L)$ of L to be the sum of the colors of the components of L . Then $H^{\varepsilon,i,j}(L) = 0$ if $\varepsilon - \text{tc}(L) = 1 \in \mathbb{Z}_2$. In particular,

$$P_L(\tau, q, t) = \tau^{\text{tc}(L)} \sum_{i,j} q^i t^j \dim H^{\text{tc}(L),i,j}(L).$$

1.3. Open problems and possible generalizations. The functoriality of the colored $\mathfrak{sl}(N)$ -homology is still open. We expect the colored $\mathfrak{sl}(N)$ -homology to be projectively functorial under colored link cobordisms. The proof should be a straightforward generalization of the projective functoriality of the Khovanov-Rozansky $\mathfrak{sl}(N)$ -homology in [18], though the algebra involved will be more complex. Strict functoriality will be a harder problem and will require more precise definitions of the morphisms involved. (Most morphisms in this paper are defined up to homotopy and scaling.) More generally, one can consider the category whose objects are knotted MOY graphs and whose morphisms are colored foam cobordisms of knotted MOY graphs. Our construction of the colored $\mathfrak{sl}(N)$ -homology actually gives an invariant of knotted MOY graphs. It seems that the colored $\mathfrak{sl}(N)$ -homology is projectively functorial on this category. Of course, to prove this, one needs to first find a precise definition and a combinatorial description of the colored foam category.

It seems that the colored $\mathfrak{sl}(N)$ -homology can be generalized in two directions:

First, we can consider a generalization of the Lee-Gornik deformation. The construction in the present paper is based on a choice of potentials induced by the polynomial z^{N+1} . It appears that the similar construction based on potentials induced by the polynomial $p(z) = z^{N+1} + \sum_{k=0}^N a_k z^k$ should also give a homological invariant for colored links. If one uses scalar coefficients $a_0, \dots, a_N \in \mathbb{C}$, then the construction will generalize [41]. If one considers a_0, \dots, a_N homogeneous variables, then this leads to a generalization of [21]. In particular, when $p(z)$ is generic, i.e. when $p'(z)$ has N distinct roots, one should be able to construct a generalization of the Lee-Gornik basis, which should lead to a further generalization of the Rasmussen invariant and a new bound for the slice genus.

Second, recall that the exterior powers of the defining representation form a set of fundamental representations of $\mathfrak{sl}(N)$. Any finite dimensional representation of $\mathfrak{sl}(N)$ can be expressed as a direct sum of tensor products of exterior powers of the defining representation. As mentioned above, the construction in the present paper gives not only a tangle invariant but also an invariant for knotted MOY graphs, which seems to contain all the algebraic information needed to construct an invariant homology for tangles colored by general finite dimensional representations of $\mathfrak{sl}(N)$. That is, using decompositions of general representations into direct sums of tensor products of exterior powers of the defining representation, we can convert a tangle colored by general representations into a collection of knotted MOY graphs, and then use algebraic operations on the chain complexes of these knotted MOY graphs to construct the homology of the original colored tangle. The construction is likely to be a generalization of [3].

1.4. Relations to other link invariants. The $\mathfrak{sl}(N)$ -homology in the present paper is closely related to several other constructions of link invariants.

Yonezawa [44] made an attempt to construct the $\mathfrak{sl}(N)$ -homology using a similar approach. First, using matrix factorizations, he defines a chain complex for knotted MOY graphs in which one of the two edges at each crossing is colored by 1. For a general knotted diagram, he splits one of the two edges at each crossing into edges colored by 1 to get a modified knotted MOY graph, in which one of the two edges at each crossing is colored by 1. He calls the chain complex of this modified knotted MOY graph the approximate complex of the original knotted MOY graph. He then establishes relations between approximate complexes of knotted

MOY graphs differed by a Reidemeister move [44, Theorem 6.6]. Yonezawa [45] further commented that, if he could somehow get a chain complex by “dividing the approximate complex by the right quantum numbers”, then, by the Krull-Schmidt property of chain complexes, the homology of this new chain complex would be invariant under Reidemeister moves and would be the $\mathfrak{sl}(N)$ -homology. But he was not able to carry out this construction. So, instead, he defines a new two-variable $\mathfrak{sl}(N)$ -polynomial invariant for colored links by dividing the Poincaré polynomial of the homology of the approximate complex by “the right quantum numbers”. It is easy to see that, up to normalization, Yonezawa’s two-variable $\mathfrak{sl}(N)$ -polynomial is the Poincaré polynomial of the $\mathfrak{sl}(N)$ -homology defined in the present paper.

Cautis and Kamnitzer [6, 7] constructed a link homology using the derived category of coherent sheaves on certain flag-like varieties. Their homology is conjectured to be isomorphic to the $\mathfrak{sl}(N)$ -Khovanov-Rozansky homology in [18]. Cautis [5] is working to generalize their homology to an $\mathfrak{sl}(N)$ -homology of links colored by exterior powers of the defining representation. We expect that the result of his construction is a homology isomorphic to the $\mathfrak{sl}(N)$ -homology constructed in the present paper.

In [27], Mackaay, Stosic and Vaz constructed a triply graded HOMFLY-PT homology for 1, 2-colored links, which generalizes the triply graded uncolored HOMFLY-PT homology defined by Khovanov and Rozansky [19]. Webster and Williamson [38] further generalized their homology to links colored by any positive integers using the equivariant cohomology of general linear groups and related spaces. It seems that Rasmussen’s spectral sequence [34] connecting the uncolored HOMFLY-PT homology to the $\mathfrak{sl}(N)$ -Khovanov-Rozansky homology should generalize to a spectral sequence connecting Webster and Williamson’s colored HOMFLY-PT homology to the colored $\mathfrak{sl}(N)$ -homology constructed in the present paper.

Kronheimer and Mrowka [22] recently defined an $SU(n)$ -homology for links using instanton gauge theory, which appears to be somewhat similar to the Lee-Gornik deformation of the $\mathfrak{sl}(N)$ -Khovanov-Rozansky homology. They expect this homology to be related to the Khovanov-Rozansky $\mathfrak{sl}(n)$ -homology by a spectral sequence. Mrowka [28] explained that their construction can be generalized to links colored by representations of $SU(n)$. It is interesting to see how this colored $SU(n)$ -homology is related to the colored $\mathfrak{sl}(N)$ -homology in the present paper.

1.5. Structure and strategy. Next we explain the structure of this paper and the strategy of our proof. We assume the reader is somewhat familiar with the works of Khovanov and Rozansky [18, 19].

Sections 2 to 4 is a review of algebraic structures used in this paper. In Section 2, we recall the definition and properties of graded matrix factorizations. Then, in Section 3, we take a closer look at graded matrix factorizations over polynomial rings. Section 4 is devoted to rings of symmetric polynomials, which serve as base rings in our construction.

Next, we define and study matrix factorizations associated to MOY graphs in Sections 5 to 10. In particular, we prove direct sum decompositions (I-V), among which decompositions (I, II, IV, V) are essential in our construction of the colored $\mathfrak{sl}(N)$ -homology.¹ Decompositions (I-IV) are generalizations of the corresponding

¹Decomposition (III) is not explicitly used in the construction of the colored $\mathfrak{sl}(N)$ -homology. The reader can skip this decomposition and its proof, that is, Subsections 7.8-7.10 and Section 8.

decompositions in [18]. We prove these four decompositions by explicitly constructing the morphisms. (Yonezawa [43] independently proved decompositions (I-III) and a special case of (IV).) Decomposition (V) is a generalization of (IV) and is far more complex. We prove it by an induction based on decomposition (IV) using the Krull-Schmidt property of graded matrix factorizations. In general, it is very hard to explicitly construct the morphisms in decomposition (V). Fortunately, we only need to use these morphisms in a few very special cases, in which the explicit forms of the morphisms are easy to come by.

The chain complex associated to a knotted MOY graph is defined in Section 11. We resolve each colored tangle diagram into a collection of MOY graphs as in [29]. Then we build a chain complex using the matrix factorizations associated to these MOY graphs in two steps. First, we compute the spaces of homotopy classes of morphisms between certain matrix factorizations. The result of this computation shows that the differential map of the chain complex exists and is unique up to homotopy and scaling, which implies that the isomorphism type of our chain complex is independent of the choices of the resolutions and markings in the construction. After that, we give an explicit construction of the differential map, which will be useful in the proof of the invariance.

The invariance of the $\mathfrak{sl}(N)$ -homology is established in Sections 12 and 13. The strategy of the proof is to use the “sliding bi-gon” method to reduce each Reidemeister move involving edges with colors greater than 1 into a sequence of Reidemeister moves involving only edges colored by 1. This strategy was first used by Murakami, Ohtsuki and Yamada in [29]. It was also used by Mackaay, Stosic, Vaz [27] and Webster, Williamson [38] to prove the invariance of the colored HOMFLY-PT homology. In Section 12, we prove a key lemma of the invariance theorem, that is, the homotopy type of our chain complex is invariant under fork sliding. Once we have this, it is easy to prove the invariance of the homotopy type of our chain complex under Reidemeister moves using an induction based on Khovanov and Rozansky’s result on the invariance in the uncolored situation [18, Section 8]. This inductive proof is given in Section 13.

Finally, we prove in Section 14 that the graded Euler characteristic of the $\mathfrak{sl}(N)$ -homology for colored links is the corresponding Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial.

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2. GRADED MATRIX FACTORIZATIONS

In this section, we review the definition and properties of graded matrix factorizations over graded \mathbb{C} -algebras, most of which can be found in [18, 19, 20, 34, 41].

Some of these properties are stated slightly more precisely here for the convenience of later applications.

2.1. \mathbb{Z} -pregraded and \mathbb{Z} -graded linear spaces. Let V be a \mathbb{C} -linear space. A \mathbb{Z} -pregrading of V is a collection $\{V^{(i)} | i \in \mathbb{Z}\}$ of \mathbb{C} -linear spaces, such that there exist injective \mathbb{C} -linear maps $\bigoplus_{i \in \mathbb{Z}} V^{(i)} \hookrightarrow V$ and $V \hookrightarrow \prod_{i \in \mathbb{Z}} V^{(i)}$ that make diagram (2.1) commutative, where the horizontal map is the standard inclusion map from the direct sum to the direct product.

$$(2.1) \quad \begin{array}{ccc} \bigoplus_{i \in \mathbb{Z}} V^{(i)} & \xrightarrow{\quad} & \prod_{i \in \mathbb{Z}} V^{(i)} \\ & \searrow \quad \nearrow & \\ & V & \end{array}$$

From now on, we will identify $V^{(i)}$ with its image in V . An element v of $V^{(i)}$ is called a homogeneous element of V of degree i . In this case, we write $\deg v = i$.

A \mathbb{Z} -pregrading $\{V^{(i)} | i \in \mathbb{Z}\}$ of V is called a \mathbb{Z} -grading if the \mathbb{C} -linear map $\bigoplus_{i \in \mathbb{Z}} V^{(i)} \hookrightarrow V$ is an isomorphism.

We say that a \mathbb{Z} -pregrading $\{V^{(i)} | i \in \mathbb{Z}\}$ of V is bounded below (resp. above) if there is an $m \in \mathbb{Z}$ such that $V^{(i)} = 0$ whenever $i < m$ (resp. $i > m$.) We call a \mathbb{Z} -pregrading bounded if it is bounded both below and above.

Let V and W be \mathbb{Z} -pregraded linear spaces with pregradings $\{V^{(i)}\}$ and $\{W^{(i)}\}$. A \mathbb{C} -linear map $f : V \rightarrow W$ is called homogeneous of degree k if and only if $f(V^{(i)}) \subset W^{(i+k)}$ for all $i \in \mathbb{Z}$.

Let V be \mathbb{Z} -pregraded linear spaces with pregrading $\{V^{(i)}\}$. For any $j \in \mathbb{Z}$, define $V\{q^j\}$ to be V with the pregrading shifted by j . That is, the pregrading $\{V\{q^j\}^{(i)}\}$ of $V\{q^j\}$ is defined by $V\{q^j\}^{(i)} = V^{(i-j)}$. More generally, for

$$F(q) = \sum_{j=k}^l a_j q^j \in \mathbb{Z}_{\geq 0}[q, q^{-1}],$$

we define the \mathbb{Z} -pregraded linear space $V\{F(q)\}$ to be

$$V\{F(q)\} = \bigoplus_{j=k}^l \underbrace{(V\{q^j\} \oplus \cdots \oplus V\{q^j\})}_{a_j\text{-fold}}$$

with the obvious pregrading $\{V\{F(q)\}^{(i)}\}$ given by

$$V\{F(q)\}^{(i)} = \bigoplus_{j=k}^l \underbrace{(V^{(i-j)} \oplus \cdots \oplus V^{(i-j)})}_{a_j\text{-fold}}$$

Note that quantum integers are elements of $\mathbb{Z}_{\geq 0}[q, q^{-1}]$. In this paper, we use the definitions

$$\begin{aligned} [j] &:= \frac{q^j - q^{-j}}{q - q^{-1}}, \\ [j]! &:= [1] \cdot [2] \cdots [j], \\ \begin{bmatrix} j \\ k \end{bmatrix} &:= \frac{[j]!}{[k]! \cdot [j-k]!}. \end{aligned}$$

It is well known that

$$(2.2) \quad \begin{bmatrix} m+n \\ n \end{bmatrix} = q^{-mn} \sum_{\lambda: l(\lambda) \leq m, \lambda_1 \leq n} q^{2|\lambda|},$$

where λ runs through partitions satisfying the given conditions. See Subsection 4.2 for definitions.

2.2. Graded modules over a graded \mathbb{C} -algebra. In the rest of this section, R will be a graded commutative unital \mathbb{C} -algebra, where “graded” means that we fix a \mathbb{Z} -grading $\{R^{(i)}\}$ on the underlying \mathbb{C} -space of R that satisfies $R^{(i)} \cdot R^{(j)} \subset R^{(i+j)}$. It is easy to see that $1 \in R^{(0)}$.

A \mathbb{Z} -grading of an R -module M is a \mathbb{Z} -grading $\{M^{(i)}\}$ of its underlying \mathbb{C} -space satisfying $R^{(i)} \cdot M^{(j)} \subset M^{(i+j)}$. A graded R -module is an R -module with a fixed \mathbb{Z} -grading. For a graded R -module M and $F(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$, $M\{F(q)\}$ is defined as above.

Lemma 2.1. *Let M_1 and M_2 be graded R -modules. Then $\text{Hom}_R(M_1, M_2)$ has a natural \mathbb{Z} -pregrading. If M_1 is finitely generated over R , then this pregrading is a grading.*

Proof. Let $\{M_1^{(i)}\}$ and $\{M_2^{(i)}\}$ be the gradings of M_1 and M_2 . Define

$$\text{Hom}_R(M_1, M_2)^{(k)} = \{f \in \text{Hom}_R(M_1, M_2) \mid f(M_1^{(i)}) \subset M_2^{(i+k)}\}.$$

We claim that $\{\text{Hom}_R(M_1, M_2)^{(k)}\}$ is a \mathbb{Z} -pregrading of $\text{Hom}_R(M_1, M_2)$. To prove this, we only need to show that:

- (i) Any $f \in \text{Hom}_R(M_1, M_2)$ can be uniquely expressed as a sum $\sum_{k=-\infty}^{\infty} f_k$, where f_k is in $\text{Hom}_R(M_1, M_2)^{(k)}$ and is called the homogeneous part of degree k of f .
- (ii) For $f, g \in \text{Hom}_R(M_1, M_2)$, $f = g$ only if all of their corresponding homogeneous part are equal.

(ii) and the uniqueness part of (i) are simple and left to the reader. We only check the existence part of (i).

For $l = 1, 2$, let $J_l^{(i)} : M_l^{(i)} \rightarrow M_l$ and $P_l^{(i)} : M_l \rightarrow M_l^{(i)}$ be the inclusion and projection in

$$M_l = \bigoplus_{i \in \mathbb{Z}} M_l^{(i)}.$$

For $k \in \mathbb{Z}$, define a \mathbb{C} -linear map $f_k : M_1 \rightarrow M_2$ by $f_k|_{M_1^{(i)}} = P_2^{(i+k)} \circ f \circ J_1^{(i)}$.

Let $m \in M_1$. Then there exist $i_1 \leq i_2$ and $j_1 \leq j_2$ such that $m \in \bigoplus_{i=i_1}^{i_2} M_1^{(i)}$ and $f(m) \in \bigoplus_{j=j_1}^{j_2} M_2^{(j)}$. It is easy to see that $f_k(m) = 0$ if $k > j_2 - i_1$ or $k < j_1 - i_2$. So the sum $\sum_{k=-\infty}^{\infty} f_k(m)$ is a finite sum for any $m \in M_1$. Thus the infinite sum $\sum_{k=-\infty}^{\infty} f_k$ is a well defined \mathbb{C} -linear map from M_1 to M_2 . One can easily see that $f = \sum_{k=-\infty}^{\infty} f_k$ as \mathbb{C} -linear maps, and that f_k is a homogeneous \mathbb{C} -linear map of degree k . It remains to check that f_k is an R -module map for every k . Let $m \in M_1^{(i)}$

and $r \in R^{(j)}$. Then $rm \in M_1^{(i+j)}$ and

$$\begin{aligned} f_k(rm) &= P_2^{(i+j+k)}(f(rm)) = P_2^{(i+j+k)}(rf(m)) \\ &= P_2^{(i+j+k)}\left(r \cdot \sum_{n=-\infty}^{\infty} f_n(m)\right) = rf_k(m). \end{aligned}$$

This implies that f_k is an R -module map and, therefore, $f_k \in \text{Hom}_R(M_1, M_2)^{(k)}$. Thus, $\{\text{Hom}_R(M_1, M_2)^{(k)}\}$ is a \mathbb{Z} -pregrading of $\text{Hom}_R(M_1, M_2)$.

Now assume that M_1 is generated by a finite subset $\{m_1, \dots, m_p\}$. For any $f \in \text{Hom}_R(M_1, M_2)$, there exist $i_1 \leq i_2, j_1 \leq j_2$ such that $m_1, \dots, m_p \in \bigoplus_{i=i_1}^{i_2} M_1^{(i)}$ and $f(m_1), \dots, f(m_p) \in \bigoplus_{j=j_1}^{j_2} M_2^{(j)}$. It follows easily that $f_k = 0$ if $k > j_2 - i_1$ or $k < j_1 - i_2$. So

$$f = \sum_{k=j_1-i_2}^{j_2-i_1} f_k \in \bigoplus_{k=-\infty}^{\infty} \text{Hom}_R(M_1, M_2)^{(k)}.$$

Thus,

$$\text{Hom}_R(M_1, M_2) = \bigoplus_{k=-\infty}^{\infty} \text{Hom}_R(M_1, M_2)^{(k)},$$

which implies that $\{\text{Hom}_R(M_1, M_2)^{(k)}\}$ is a \mathbb{Z} -grading of $\text{Hom}_R(M_1, M_2)$. \square

In the present paper, we are especially interested in free graded modules over R . Note that a free graded module need not have a basis consisting of homogeneous elements. Following [30, Chapter 13], we introduce the following definition.

Definition 2.2. A graded module M over R is called graded-free if and only if it is a free module over R with a homogeneous basis.

All the modules involved in the construction of the $\mathfrak{sl}(N)$ -homology are modules over polynomial rings. We will see in Section 3 that, if the grading of a free graded module over a polynomial ring is bounded below, then this module is graded-free.

2.3. Graded matrix factorizations. Recall that, $N (\geq 2)$ is a fixed integer throughout the present paper. (It is the “ N ” in “ $\mathfrak{sl}(N)$ ”.)

Let R be a graded commutative unital \mathbb{C} -algebra, and w a homogeneous element of R of degree $2N + 2$. A graded matrix factorization M over R with potential w is a collection of two free graded R -modules M_0, M_1 and two homogeneous R -module maps $d_0 : M_0 \rightarrow M_1, d_1 : M_1 \rightarrow M_0$ of degree $N + 1$, called differential maps, s.t.

$$d_1 \circ d_0 = w \cdot \text{id}_{M_0}, \quad d_0 \circ d_1 = w \cdot \text{id}_{M_1}.$$

We usually write M as

$$M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0.$$

M has two gradings: a \mathbb{Z}_2 -grading that takes value ε on M_ε and a quantum grading, which is the \mathbb{Z} -grading of the underlying graded R -module. We denote the degree from the \mathbb{Z}_2 -grading by “ $\deg_{\mathbb{Z}_2}$ ” and the degree from the quantum grading by “ \deg ”.

Following [18], we denote by $M \langle 1 \rangle$ the matrix factorization

$$M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1,$$

and write $M \langle j \rangle = M \underbrace{\langle 1 \rangle \cdots \langle 1 \rangle}_{j \text{ times}}$.

For graded matrix factorizations M with potential w and M' with potential w' , the tensor product $M \otimes M'$ is the graded matrix factorization with

$$\begin{aligned} (M \otimes M')_0 &= (M_0 \otimes M'_0) \oplus (M_1 \otimes M'_1), \\ (M \otimes M')_1 &= (M_1 \otimes M'_0) \oplus (M_0 \otimes M'_1), \end{aligned}$$

and the differential given by signed Leibniz rule, i.e., for $m \in M_\varepsilon$ and $m' \in M'$,

$$d(m \otimes m') = (dm) \otimes m' + (-1)^\varepsilon m \otimes (dm').$$

The potential of $M \otimes M'$ is $w + w'$.

Definition 2.3. If $a_0, a_1 \in R$ are homogeneous elements with $\deg a_0 + \deg a_1 = 2N+2$, then denote by $(a_0, a_1)_R$ the matrix factorization $R \xrightarrow{a_0} R\{q^{N+1-\deg a_0}\} \xrightarrow{a_1} R$, which has potential $a_0 a_1$. More generally, if $a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1} \in R$ are homogeneous with $\deg a_{j,0} + \deg a_{j,1} = 2N+2$, denote by

$$\begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_R$$

the tensor product

$$(a_{1,0}, a_{1,1})_R \otimes_R (a_{2,0}, a_{2,1})_R \otimes_R \dots \otimes_R (a_{k,0}, a_{k,1})_R,$$

which is a graded matrix factorization with potential $\sum_{j=1}^k a_{j,0} a_{j,1}$, and is call the Koszul matrix factorization associated to the above matrix. We drop “ R ” the notation when it is clear from the context. Note that the above Koszul matrix factorization is finitely generated over R .

Since the Koszul matrix factorizations we use in this paper are more complex than those in [18, 19, 34, 41], it is generally harder to compute them. So it is more important to keep good track of the signs. For this reason, we introduce the following notations.

Definition 2.4. • Let $I = \{0, 1\}$. Define $\bar{1} = 0$ and $\bar{0} = 1$.

- For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$, define $|\varepsilon| = \sum_{j=1}^k \varepsilon_j$, and for $1 \leq i \leq k$, define $|\varepsilon|_i = \sum_{j=1}^{i-1} \varepsilon_j$. Also define $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_k)$ and $\varepsilon' = (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_1)$.
- In $(a_0, a_1)_R$, denote by 1_0 the unit element of the copy of R with \mathbb{Z}_2 -grading 0, and by 1_1 the unit element of the copy of R with \mathbb{Z}_2 -grading 1. Note that $\{1_0, 1_1\}$ is an R -basis for $(a_0, a_1)_R$.
- In

$$M = \begin{pmatrix} a_{1,0} & a_{1,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_R,$$

for any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$, define $1_\varepsilon = 1_{\varepsilon_1} \otimes \dots \otimes 1_{\varepsilon_k}$ in the tensor product

$$(a_{1,0}, a_{1,1})_R \otimes_R \dots \otimes_R (a_{k,0}, a_{k,1})_R.$$

Note that $\{1_\varepsilon \mid \varepsilon \in I^k\}$ is an R -basis for M , and 1_ε is a homogeneous element with \mathbb{Z}_2 -degree $|\varepsilon|$ and quantum degree $\sum_{j=1}^k \varepsilon_j(N+1-\deg a_{j,0})$.

In the above notations, the differential of M is given by

$$d(1_\varepsilon) = \sum_{j=1}^k (-1)^{|\varepsilon|_j} a_{j,\varepsilon_j} \cdot 1_{(\varepsilon_1, \dots, \varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)}.$$

Remark 2.5. In many cases, only the parity of $|\varepsilon|$ matters and I can be viewed as \mathbb{Z}_2 . But, in some situations, we need more information, and, thus, I can not be identified with \mathbb{Z}_2 .

2.4. Morphisms of graded matrix factorizations. Given two graded matrix factorizations M with potential w and M' with potential w' over R , consider the R -module $\text{Hom}_R(M, M')$. It admits a \mathbb{Z}_2 -grading that takes value

$$\begin{cases} 0 & \text{on } \text{Hom}_R^0(M, M') = \text{Hom}_R(M_0, M'_0) \oplus \text{Hom}_R(M_1, M'_1), \\ 1 & \text{on } \text{Hom}_R^1(M, M') = \text{Hom}_R(M_1, M'_0) \oplus \text{Hom}_R(M_0, M'_1). \end{cases}$$

By Lemma 2.1, it also admits a quantum pregrading induced by the quantum gradings of homogeneous elements. Moreover, $\text{Hom}_R(M, M')$ has a differential map d given by

$$d(f) = d_{M'} \circ f - (-1)^\varepsilon f \circ d_M \text{ for } f \in \text{Hom}_R^\varepsilon(M, M').$$

Note that d is homogeneous of degree $N + 1$ and satisfies that

$$d^2 = (w' - w) \cdot \text{id}_{\text{Hom}_R(M, M')}.$$

Following [18], we write $M_\bullet = \text{Hom}_R(M, R)$.

In general, $\text{Hom}_R(M, M')$ is not a graded matrix factorization since $\text{Hom}_R(M, M')$ might not be a free module and the quantum pregrading might not be a grading. But we have the following easy lemma.

Lemma 2.6. *Let M and M' be as above. Assume that M is finitely generated over R . Then $\text{Hom}_R(M, M')$ is a graded matrix factorization over R of potential $w' - w$. In particular, $M_\bullet = \text{Hom}_R(M, R)$ is a graded matrix factorization over R of potential $-w$.*

Proof. Since M is finitely generated, we know that $\text{Hom}_R(M, M')$ is a free R -module and, by Lemma 2.1, the quantum pregrading is a grading. \square

Now let M and M' be two graded matrix factorizations over R with potential w . Then $\text{Hom}_R(M, M')$, with the above differential map d , is a cyclic chain complex. We say that an R -module map $f : M \rightarrow M'$ is a morphism of matrix factorizations if and only if $df = 0$. Or, equivalently, for $f \in \text{Hom}_R^\varepsilon(M, M')$, f is a morphism of matrix factorizations if and only if $d_{M'} \circ f = (-1)^\varepsilon f \circ d_M$. f is called an isomorphism of matrix factorizations if it is a morphism of matrix factorizations and an isomorphism of the underlying R -modules. Two morphisms $f, g : M \rightarrow M'$ of \mathbb{Z}_2 -degree ε are homotopic if $f - g$ is a boundary element in $\text{Hom}_R(M, M')$, that is, if $\exists h \in \text{Hom}_R^{\varepsilon+1}(M, M')$ such that $f - g = d(h) = d_{M'} \circ h - (-1)^{\varepsilon+1} h \circ d_M$.

Definition 2.7. Let M and M' be two graded matrix factorizations over R with the same potential.

- (a) We say that M, M' are isomorphic, or $M \cong M'$, if and only if there is a homogeneous isomorphism $f : M \rightarrow M'$ that preserves both gradings.
- (b) We say that M, M' are homotopic, or $M \simeq M'$, if and only if there are homogeneous morphisms $f : M \rightarrow M'$ and $g : M' \rightarrow M$ preserving both gradings such that $g \circ f \simeq \text{id}_M$ and $f \circ g \simeq \text{id}_{M'}$.

Lemma 2.8. *Let M and M' be two graded matrix factorizations over R with potentials w and w' . Assume that M is finitely generated over R . Then the natural R -module isomorphism*

$$M' \otimes M_\bullet = M' \otimes \text{Hom}_R(M, R) \xrightarrow{\cong} \text{Hom}_R(M, M')$$

is a homogeneous isomorphism preserving both gradings.

In particular, $\text{Hom}_R(M, M') \cong M' \otimes M_\bullet$ as graded matrix factorizations.

Proof. By Lemma 2.6, $M' \otimes M_\bullet$ and $\text{Hom}_R(M, M')$ are both graded matrix factorizations over R with potential $w' - w$. The natural isomorphism F between them is given by $F(m' \otimes f)(m) = f(m) \cdot m'$ for all $m' \in M'$, $f \in M_\bullet$ and $m \in M$. It is easy to check that F preserves both gradings and commutes with the differential maps. \square

The following lemma specifies the sign convention we use when tensoring two morphisms of matrix factorizations.

Lemma 2.9. *Let R be a graded commutative unital \mathbb{C} -algebra, and $M, M', \mathcal{M}, \mathcal{M}'$ graded matrix factorizations over R such that M, \mathcal{M} have the same potential and M', \mathcal{M}' have the same potential. Assume that $f : M \rightarrow \mathcal{M}$ and $f' : M' \rightarrow \mathcal{M}'$ are morphisms of matrix factorizations of \mathbb{Z}_2 -degrees j and j' . Define $F : M \otimes M' \rightarrow \mathcal{M} \otimes \mathcal{M}'$ by $F(m \otimes m') = (-1)^{i \cdot j'} f(m) \otimes f'(m')$ for $m \in M_i$ and $m' \in M'$. Then F is a morphism of matrix factorizations of \mathbb{Z}_2 -degree $j + j'$. In particular, if f or f' is homotopic to 0, then so is F .*

From now on, we will write $F = f \otimes f'$.

Proof.

$$\begin{aligned} F \circ d(m \otimes m') &= F((dm) \otimes m' + (-1)^i m \otimes (dm')) \\ &= (-1)^{(i+1)j'} f(dm) \otimes f'(m') + (-1)^{i+j'} f(m) \otimes f'(dm'), \end{aligned}$$

$$\begin{aligned} d \circ F(m \otimes m') &= (-1)^{ij'} d(f(m) \otimes f'(m')) \\ &= (-1)^{ij'} (d(f(m)) \otimes f'(m') + (-1)^{i+j} f(m) \otimes d(f'(m'))) \\ &= (-1)^{ij'+j} f(dm) \otimes f'(m') + (-1)^{ij'+i+j+j'} f(m) \otimes f'(dm'). \end{aligned}$$

So $F \circ d = (-1)^{j+j'} d \circ F$, i.e. F is a morphism of matrix factorizations of \mathbb{Z}_2 -degree $j + j'$.

If f is homotopic to 0. Then there exists $h \in \text{Hom}_R^{j+1}(M, \mathcal{M})$ such that

$$f = d(h) = d \circ h - (-1)^{j+1} h \circ d.$$

Define $H \in \text{Hom}_R^{j+j'+1}(M \otimes M', \mathcal{M} \otimes \mathcal{M}')$ by

$$H(m \otimes m') := (-1)^{ij'} h(m) \otimes f'(m'), \text{ for } m \in M_i \text{ and } m' \in M'.$$

Then $d(H) = d \circ H - (-1)^{j+j'+1} H \circ d = F$. So F is homotopic to 0.

If f' is homotopic to 0. Then there exists $h' \in \text{Hom}_R^{j'+1}(M', \mathcal{M}')$ such that

$$f' = d(h') = d \circ h' - (-1)^{j'+1} h' \circ d.$$

Define $H' \in \text{Hom}_R^{j+j'+1}(M \otimes M', \mathcal{M} \otimes \mathcal{M}')$ by

$$H'(m \otimes m') := (-1)^{i(j'+1)} f(m) \otimes h'(m'), \text{ for } m \in M_i \text{ and } m' \in M'.$$

Then $d(H') = d \circ H' - (-1)^{j+j'+1} H' \circ d = (-1)^j F$. So F is homotopic to 0. \square

The following lemma is [18, Proposition 2].

Lemma 2.10. [18, Proposition 2] *Let R be a graded commutative unital \mathbb{C} -algebra, and $a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1}$ homogeneous elements of R with $\deg a_{j,0} + \deg a_{j,1} = 2N + 2 \forall j$. Let*

$$M = \begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_R$$

If x is an element of the ideal $(a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1})$ of R , then multiplication by x is a null-homotopic endomorphism of M .

Proof. (Following [18].) Multiplications by $a_{i,0}$ and $a_{i,1}$ are null-homotopic endomorphisms of $(a_{1,0}, a_{1,1})$, and therefore, by Lemma 2.9, are null-homotopic endomorphisms of M . \square

Next we give precise definitions of several isomorphisms used in [18], which allow us to keep track of signs in later applications.

Lemma 2.11. *Let R be a graded commutative unital \mathbb{C} -algebra, and $a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1}$ homogeneous elements of R with $\deg a_{j,0} + \deg a_{j,1} = 2N + 2 \forall j$. Let*

$$M = \begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_R \quad \text{and} \quad M' = \begin{pmatrix} -a_{k,1} & a_{k,0} \\ -a_{k-1,1} & a_{k-1,0} \\ \dots & \dots \\ -a_{1,1} & a_{1,0} \end{pmatrix}_R.$$

Denote by $\{1_\varepsilon^ \mid \varepsilon \in I^k\}$ the basis of M_\bullet dual to $\{1_\varepsilon \mid \varepsilon \in I^k\}$, i.e. $1_\varepsilon^*(1_\varepsilon) = 1$ and $1_\varepsilon^*(1_\sigma) = 0$ if $\sigma \neq \varepsilon$. Then the R -homomorphism $F : M_\bullet \rightarrow M'$ given by $F(1_\varepsilon^*) = 1_{\varepsilon'}$ is an isomorphism of matrix factorizations that preserves both gradings.*

Proof. F is clearly an isomorphism of R -modules. Recall that $\varepsilon' = (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_1)$ if $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) \in I^k$. The element 1_ε^* of M_\bullet has \mathbb{Z}_2 -grading $|\varepsilon|$ and quantum grading $-\sum_{j=1}^k \varepsilon_j(N+1 - \deg a_{j,0}) = \sum_{j=1}^k \varepsilon_j(N+1 - \deg a_{j,1})$. And the element $1_{\varepsilon'}$ of M' has \mathbb{Z}_2 -grading $|\varepsilon'| = |\varepsilon|$ and quantum grading $\sum_{j=1}^k \varepsilon_j(N+1 - \deg a_{j,1})$. So F preserves both gradings. It remains to show that F is a morphism of matrix factorizations. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$, a straightforward calculation shows that

$$d(1_\varepsilon^*) = \sum_{j=1}^k (-1)^{|\varepsilon| - |\varepsilon|_j + 1} a_{j, \overline{\varepsilon_j}} \cdot 1_{(\varepsilon_1, \dots, \varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)},$$

$$d(1_{\varepsilon'}) = \sum_{j=1}^k (-1)^{|\varepsilon| - |\varepsilon|_j + 1} a_{j, \overline{\varepsilon_j}} \cdot 1_{(\varepsilon_k, \dots, \varepsilon_{j+1}, \overline{\varepsilon_j}, \varepsilon_{j-1}, \dots, \varepsilon_1)}.$$

So $d_{M'} \circ F = F \circ d_{M_\bullet}$. \square

Lemma 2.12. *Let R be a graded commutative unital \mathbb{C} -algebra, and $a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1}$ homogeneous elements of R with $\deg a_{j,0} + \deg a_{j,1} = 2N + 2 \forall j$. Let*

$$M = \begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_R \quad \text{and} \quad M' = \begin{pmatrix} a_{k,0} & a_{k,1} \\ a_{k-1,0} & a_{k-1,1} \\ \dots & \dots \\ a_{1,0} & a_{1,1} \end{pmatrix}_R.$$

Define an R -homomorphism $F : M \rightarrow M'$ by $F(1_\varepsilon) = (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2}} 1_\varepsilon$, $\forall \varepsilon \in I^k$. Then F is an isomorphism of matrix factorizations that preserves both gradings.

Proof. It is clear that F is an isomorphism of R -modules and preserves both gradings. It remains to show that F is a morphism of matrix factorizations. When $k = 1$, there is nothing to prove. When $k = 2$, F is given by the following diagram

$$\begin{array}{ccccc} \begin{pmatrix} R \cdot 1_{(0,0)} \\ R \cdot 1_{(1,1)} \end{pmatrix} & \xrightarrow{\begin{pmatrix} a_{1,0} & -a_{2,1} \\ a_{2,0} & a_{1,1} \end{pmatrix}} & \begin{pmatrix} R \cdot 1_{(1,0)} \\ R \cdot 1_{(0,1)} \end{pmatrix} & \xrightarrow{\begin{pmatrix} a_{1,1} & a_{2,1} \\ -a_{2,0} & a_{1,0} \end{pmatrix}} & \begin{pmatrix} R \cdot 1_{(0,0)} \\ R \cdot 1_{(1,1)} \end{pmatrix} \\ \downarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} R \cdot 1_{(0,0)} \\ R \cdot 1_{(1,1)} \end{pmatrix} & \xrightarrow{\begin{pmatrix} a_{2,0} & -a_{1,1} \\ a_{1,0} & a_{2,1} \end{pmatrix}} & \begin{pmatrix} R \cdot 1_{(1,0)} \\ R \cdot 1_{(0,1)} \end{pmatrix} & \xrightarrow{\begin{pmatrix} a_{2,1} & a_{1,1} \\ -a_{1,0} & a_{2,0} \end{pmatrix}} & \begin{pmatrix} R \cdot 1_{(0,0)} \\ R \cdot 1_{(1,1)} \end{pmatrix} \end{array}$$

where the first row is M , the second row is M' , and F is given by the vertical arrows. A simple direct computation shows that F is a morphism. The general $k \geq 2$ case follows from the $k = 2$ case by a straightforward induction using Lemma 2.9. \square

Lemma 2.13. *Let R be a graded commutative unital \mathbb{C} -algebra, and $a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1}$ homogeneous elements of R with $\deg a_{j,0} + \deg a_{j,1} = 2N + 2 \forall j$. Let*

$$M = \begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_R \quad \text{and} \quad M' = \begin{pmatrix} a_{1,1} & a_{1,0} \\ a_{2,1} & a_{2,0} \\ \dots & \dots \\ a_{k,1} & a_{k,0} \end{pmatrix}_R.$$

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$, write $s(\varepsilon) = \sum_{j=1}^{k-1} (k-j)\varepsilon_j$. Define an R -homomorphism $F : M \rightarrow M'$ by $F(1_\varepsilon) = (-1)^{|\varepsilon|+s(\varepsilon)} 1_\varepsilon \forall \varepsilon \in I^k$. Then F is an isomorphism of matrix factorizations of \mathbb{Z}_2 -degree k and quantum degree $\sum_{j=1}^k (N+1 - \deg a_{j,1})$.

Proof. F is clearly an isomorphism of R -modules. And the claims about its two gradings are easy to verify. Only need to check that F is a morphism of matrix factorization. This is again easy when $k = 1$. The general $k \geq 1$ case follows from the $k = 1$ case by a straightforward induction using Lemma 2.9. \square

2.5. Elementary operations on Koszul matrix factorizations. Khovanov and Rozansky [18, 19] and Rasmussen [34] introduced several elementary operations on Koszul matrix factorizations that give isomorphic or homotopic graded matrix factorizations. In this subsection, we recall these operations.

Lemma 2.14. [34, 41] *Let M be the graded matrix factorization*

$$M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0.$$

over R with potential w . Suppose that $H_i : M_i \rightarrow M_i$ are graded homomorphisms with $H_i^2 = 0$. Define $\tilde{d}_i : M_i \rightarrow M_{i+1}$ by

$$\tilde{d}_i = (\text{id}_{M_{i+1}} - H_{i+1}) \circ d_i \circ (\text{id}_{M_i} + H_i),$$

and \tilde{M} by

$$M_0 \xrightarrow{\tilde{d}_0} M_1 \xrightarrow{\tilde{d}_1} M_0.$$

Then \widetilde{M} is also a graded matrix factorization over R with potential w . And $M \cong \widetilde{M}$.

Corollary 2.15. [34] Suppose $a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}, k$ are homogeneous elements in R satisfying $\deg a_{j,0} + \deg a_{j,1} = 2N + 2$ and $\deg k = \deg a_{1,0} + \deg a_{2,0} - 2N - 2$. Then

$$\begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \end{pmatrix}_R \cong \begin{pmatrix} a_{1,0} + ka_{2,1} & a_{1,1} \\ a_{2,0} - ka_{1,1} & a_{2,1} \end{pmatrix}_R.$$

Corollary 2.16. [18, 34] Suppose $a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}, c$ are homogeneous elements in R satisfying $\deg a_{j,0} + \deg a_{j,1} = 2N + 2$ and $\deg c = \deg a_{1,0} - \deg a_{2,0}$. Then

$$\begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \end{pmatrix}_R \cong \begin{pmatrix} a_{1,0} + ca_{2,0} & a_{1,1} \\ a_{2,0} & a_{2,1} - ca_{1,1} \end{pmatrix}_R.$$

The proofs of the above can be found in [18, 19, 34, 41] and are omitted.

Definition 2.17. Let R be a commutative ring, and $a_1, \dots, a_k \in R$. The sequence $\{a_1, \dots, a_k\}$ is called R -regular if a_1 is not a zero divisor in R and a_j is not a zero divisor in $R/(a_1, \dots, a_{j-1})$ for $j = 2, \dots, k$.

The next lemma is [20, Theorem 2.1] and a generalization of [34, Lemma 3.10].

Lemma 2.18. [20, 34] Let R be a graded commutative unital \mathbb{C} -algebra. Suppose that $\{a_1, \dots, a_k\}$ is an R -regular sequence of homogeneous elements of R with $\deg a_j \leq 2N + 2 \ \forall \ j = 1, \dots, k$. Assume that $f_1, \dots, f_k, g_1, \dots, g_k$ are homogeneous elements of R such that $\deg f_j = \deg g_j = 2N + 2 - \deg a_j$ and $\sum_{j=1}^k f_j a_j = \sum_{j=1}^k g_j a_j$. Then

$$\begin{pmatrix} f_1 & a_1 \\ \dots & \dots \\ f_k & a_k \end{pmatrix}_R \cong \begin{pmatrix} g_1 & a_1 \\ \dots & \dots \\ g_k & a_k \end{pmatrix}_R.$$

Proof. Induct on k . If $k = 1$, then a_1 is not a zero divisor in R and $(f_1 - g_1)a_1 = 0$. So $f_1 = g_1$ and $(f_1, a_1)_R = (g_1, a_1)_R$. Assume that the lemma is true for $k = m$. Consider the case $k = m + 1$. a_{m+1} is not a zero divisor in $R/(a_1, \dots, a_m)$. But

$$(f_{m+1} - g_{m+1})a_{m+1} = \sum_{j=1}^m (g_j - f_j)a_j \in (a_1, \dots, a_m).$$

So $f_{m+1} - g_{m+1} \in (a_1, \dots, a_m)$, i.e. there exist $c_1, \dots, c_m \in R$ such that

$$f_{m+1} - g_{m+1} = \sum_{j=1}^m c_j a_j.$$

Thus, by Corollary 2.15,

$$\begin{pmatrix} f_1 & a_1 \\ \dots & \dots \\ f_m & a_m \\ f_{m+1} & a_{m+1} \end{pmatrix}_R \cong \begin{pmatrix} f_1 + c_1 a_{m+1} & a_1 \\ \dots & \dots \\ f_m + c_m a_{m+1} & a_m \\ g_{m+1} & a_{m+1} \end{pmatrix}_R.$$

It is easy to see that

$$\sum_{j=1}^m (f_j + c_j a_{m+1})a_j = \sum_{j=1}^m g_j a_j.$$

By induction hypothesis,

$$\begin{pmatrix} f_1 + c_1 a_{m+1}, & a_1 \\ \dots & \dots \\ f_m + c_m a_{m+1}, & a_m \end{pmatrix}_R \cong \begin{pmatrix} g_1, & a_1 \\ \dots & \dots \\ g_m, & a_m \end{pmatrix}_R.$$

Therefore,

$$\begin{pmatrix} f_1, & a_1 \\ \dots & \dots \\ f_m, & a_m \\ f_{m+1}, & a_{m+1} \end{pmatrix}_R \cong \begin{pmatrix} f_1 + c_1 a_{m+1}, & a_1 \\ \dots & \dots \\ f_m + c_m a_{m+1}, & a_m \\ g_{m+1}, & a_{m+1} \end{pmatrix}_R \cong \begin{pmatrix} g_1, & a_1 \\ \dots & \dots \\ g_m, & a_m \\ g_{m+1}, & a_{m+1} \end{pmatrix}_R.$$

□

Next we give three versions of [18, Proposition 9], which give a method of simplifying matrix factorizations. Their proofs also give a method of finding cycles representing a given homology class in some chain complexes and finding morphisms of matrix factorizations representing a given homotopy class, which is important for our purpose. So we give their full proofs here.

Proposition 2.19 (strong version). *Let R be a graded commutative unital \mathbb{C} -algebra, and x a homogeneous indeterminant with $\deg x \leq 2N+2$. Let $P : R[x] \rightarrow R$ be the evaluation map at $x = 0$, i.e. $P(f(x)) = f(0) \forall f(x) \in R[x]$.*

Suppose that $a_1, \dots, a_k, b_1, \dots, b_k$ are homogeneous elements of $R[x]$ such that

- $\deg a_j + \deg b_j = 2N + 2 \forall j = 1, \dots, k$,
- $\sum_{j=1}^k a_j b_j \in R$,
- $\exists i \in \{1, \dots, k\}$ such that $b_i = x$.

Then

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \dots & \dots \\ a_k & b_k \end{pmatrix}_{R[x]} \quad \text{and} \quad M' = \begin{pmatrix} P(a_1) & P(b_1) \\ P(a_2) & P(b_2) \\ \dots & \dots \\ P(a_{i-1}) & P(b_{i-1}) \\ P(a_{i+1}) & P(b_{i+1}) \\ \dots & \dots \\ P(a_k) & P(b_k) \end{pmatrix}_R$$

are homotopic as graded matrix factorizations over R .

Proof. For $j \neq i$, Write $a'_j = P(a_j) \in R$ and $b'_j = P(b_j) \in R$. Then $\exists! c_j, k_j \in R[x]$ such that $a_j = a'_j + k_j x$ and $b_j = b'_j + c_j x$. By Corollaries 2.15 and 2.16,

$$M \cong M'' := \begin{pmatrix} a'_1 & b'_1 \\ \dots & \dots \\ a'_{i-1} & b'_{i-1} \\ a & x \\ a'_{i+1} & b'_{i+1} \\ \dots & \dots \\ a'_k & b'_k \end{pmatrix}_{R[x]},$$

where $a = a_i + \sum_{j \neq i} k_j b_j + \sum_{j \neq i} c_j a'_j$. Since M, M'' have the same potential, we know that $ax = \sum_{j=1}^k a_j b_j - \sum_{j \neq i} a'_j b'_j \in R$. So $a = 0$. Thus,

$$M'' = \begin{pmatrix} a'_1 & b'_1 \\ \vdots & \vdots \\ a'_{i-1} & b'_{i-1} \\ 0 & x \\ a'_{i+1} & b'_{i+1} \\ \vdots & \vdots \\ a'_k & b'_k \end{pmatrix}_{R[x]}.$$

Define R -module homomorphism $F : M'' \rightarrow M'$ by

$$F(f(x)1_\varepsilon) = \begin{cases} f(0)1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_k)} & \text{if } \varepsilon_i = 0, \\ 0 & \text{if } \varepsilon_i = 1, \end{cases}$$

for $f(x) \in R[x]$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$. And define R -module homomorphism $G : M' \rightarrow M''$ by

$$G(r1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_k)}) = r1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \dots, \varepsilon_k)}$$

for $r \in R$ and $(\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_k) \in I^{k-1}$.

One can easily check that F and G are morphisms of matrix factorizations preserving both gradings and $F \circ G = \text{id}_{M'}$. Note that $M'' = \ker F \oplus \text{Im} G$ and

$$\begin{aligned} G \circ F|_{\ker F} &= 0, \\ G \circ F|_{\text{Im} G} &= \text{id}_{\text{Im} G}. \end{aligned}$$

Define an R -module homomorphism $h : M'' \rightarrow M''$ by

$$\begin{aligned} h(1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_k)}) &= 0, \\ h((r + xf(x))1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \dots, \varepsilon_k)}) &= (-1)^{\sum_{j=1}^{i-1} \varepsilon_j} f(x)1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_k)} \end{aligned}$$

for $r \in R, f(x) \in R[x]$ and $\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_k \in I$. A straightforward computation shows that

$$\begin{aligned} (d \circ h + h \circ d)|_{\ker F} &= \text{id}_{\ker F}, \\ (d \circ h + h \circ d)|_{\text{Im} G} &= 0. \end{aligned}$$

So $\text{id}_{M''} - G \circ F = d \circ h + h \circ d$. Thus, we have $M'' \simeq M'$ and, therefore, $M \simeq M'$ as matrix factorizations over R . \square

Proposition 2.20 (weak version). *Let R be a graded commutative unital \mathbb{C} -algebra, and $a_1, \dots, a_k, b_1, \dots, b_k$ homogeneous elements of R such that $\deg a_j + \deg b_j = 2N + 2$ and $\sum_{j=1}^k a_j b_j = 0$. Then the matrix factorization*

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix}_R$$

is a cyclic chain complex. Assume that, for a given $i \in \{1, \dots, k\}$, b_i is not a zero divisor in R . Define $R' = R/(b_i)$, which inherits the grading of R . Let $P : R \rightarrow R'$

be the standard projection. Then

$$M' = \begin{pmatrix} P(a_1) & P(b_1) \\ P(a_2) & P(b_2) \\ \dots & \dots \\ P(a_{i-1}) & P(b_{i-1}) \\ P(a_{i+1}) & P(b_{i+1}) \\ \dots & \dots \\ P(a_k) & P(b_k) \end{pmatrix}_{R'}$$

is also a chain complex. And $H(M) \cong H(M')$ as graded R -modules.

Proof. Define an R -module homomorphism $F : M \rightarrow M'$ by

$$F(r1_\varepsilon) = \begin{cases} P(r)1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_k)} & \text{if } \varepsilon_i = 0, \\ 0 & \text{if } \varepsilon_i = 1, \end{cases}$$

for $r \in R$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$. It is easy to check that F is a surjective morphism of matrix factorizations preserving both gradings. The kernel of F is the subcomplex

$$\ker F = \bigoplus_{\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_k \in I} (R \cdot 1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_k)} \oplus b_i R \cdot 1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \dots, \varepsilon_k)}).$$

Since b_i is not a zero divisor, the division map $\varphi : b_i R \rightarrow R$ given by $\varphi(b_i r) = r$ is well defined. Define an R -module homomorphism $h : \ker F \rightarrow \ker F$ by

$$\begin{aligned} h(1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_k)}) &= 0, \\ h(b_i 1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \dots, \varepsilon_k)}) &= (-1)^{\sum_{j=1}^{i-1} \varepsilon_j} 1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_{i+1}, \dots, \varepsilon_k)}. \end{aligned}$$

Then

$$d|_{\ker F} \circ h + h \circ d|_{\ker F} = \text{id}_{\ker F},$$

where d is the differential map of M . In particular, this means that $H(\ker F) = 0$. Then, using the long exact sequence induced by

$$0 \rightarrow \ker F \rightarrow M \xrightarrow{F} M' \rightarrow 0,$$

it is easy to see that F is a quasi-isomorphism. \square

Remark 2.21. The above proof of Proposition 2.20 also gives a method of finding cycles in M whose image under F is a given cycle in M' . Indeed, for every cycle α in M' , one can find an element $\beta \in M$ such that $F(\beta) = \alpha$. Then $F(d\beta) = d'F(\beta) = d'\alpha = 0$, where d' is the differential map of M' . So $d\beta \in \ker F$ and $d\beta = dh(d\beta) + hd(d\beta) = dh(d\beta)$. Thus, $\beta - h(d\beta)$ is a cycle in M . By definition, it clear that $F \circ h = 0$. So $F(\beta - h(d\beta)) = \alpha$. This observation is useful in finding cycles representing a given homology class and morphisms representing a given homotopy class. (In the situation in Proposition 2.19, one can also do the same by explicitly computing the morphism $M' \xrightarrow{\cong} M'' \xrightarrow{\cong} M$, which is usually not any easier in practice.) This method also applies to the situation in corollaries 2.24 and 2.25, i.e. contracting the matrix factorization using an entry in the left column.

Next we give the dual version of Proposition 2.20.

Corollary 2.22 (dual version). *Let R be a graded commutative unital \mathbb{C} -algebra, and \hat{R} a graded commutative unital sub-algebra of R such that R is a free \hat{R} -module. Suppose that $a_1, \dots, a_k, b_1, \dots, b_k$ are homogeneous elements of R such that $\deg a_j + \deg b_j = 2N + 2$ and $\sum_{j=1}^k a_j b_j = w \in \hat{R}$. Assume that, for a given $i \in \{1, \dots, k\}$, b_i is not a zero divisor in R and $R' = R/(b_i)$ is also a free \hat{R} -module. Define*

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \dots & \dots \\ a_k & b_k \end{pmatrix}_R$$

and

$$M' = \begin{pmatrix} P(a_1) & P(b_1) \\ P(a_2) & P(b_2) \\ \dots & \dots \\ P(a_{i-1}) & P(b_{i-1}) \\ P(a_{i+1}) & P(b_{i+1}) \\ \dots & \dots \\ P(a_k) & P(b_k) \end{pmatrix}_{R'}$$

where $P : R \rightarrow R'$ is the standard projection. Then, for any matrix factorization M'' over \hat{R} with potential w , there is an quasi-isomorphism

$$\mathrm{Hom}_{\hat{R}}(M', M'') \rightarrow \mathrm{Hom}_{\hat{R}}(M, M'')$$

preserving both gradings.

Proof. Define an R -module homomorphism $F : M \rightarrow M'$ by

$$F(r1_\varepsilon) = \begin{cases} P(r)1_{(\varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_k)} & \text{if } \varepsilon_i = 0, \\ 0 & \text{if } \varepsilon_i = 1, \end{cases}$$

for $r \in R$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in I^k$. Then F is a surjective morphism of matrix factorizations preserving both gradings. So we have a short exact sequence

$$0 \rightarrow \ker F \rightarrow M \xrightarrow{F} M' \rightarrow 0.$$

Note that $\ker F$ and M are free R -modules and M' is a free R' -module. Thus, the above is a short exact sequence of free \hat{R} -modules. This implies that

$$0 \rightarrow \mathrm{Hom}_{\hat{R}}(M', M'') \xrightarrow{F^\sharp} \mathrm{Hom}_{\hat{R}}(M, M'') \rightarrow \mathrm{Hom}_{\hat{R}}(\ker F, M'') \rightarrow 0$$

is also exact. Recall that there exists $h : \ker F \rightarrow \ker F$ of \mathbb{Z}_2 -degree 1 such that $\mathrm{id}_{\ker F} = d_M|_{\ker F} \circ h + h \circ d_M|_{\ker F}$. Define

$$H : \mathrm{Hom}_{\hat{R}}(\ker F, M'') \rightarrow \mathrm{Hom}_{\hat{R}}(\ker F, M'')$$

by $H(f) = (-1)^j f \circ h$ if f has \mathbb{Z}_2 -degree j . H has \mathbb{Z}_2 -degree 1. For $f \in \mathrm{Hom}_{\hat{R}}(\ker F, M'')$ of \mathbb{Z}_2 -degree j ,

$$\begin{aligned} & (d \circ H + H \circ d)(f) \\ &= d(H(f)) + H(d(f)) \\ &= (-1)^j d(f \circ h) + (-1)^{j+1} (df) \circ h \\ &= (-1)^j (d_{M''} \circ f \circ h - (-1)^{j+1} f \circ h \circ d_M|_{\ker F}) + (-1)^{j+1} (d_{M''} \circ f \circ h - (-1)^j f \circ d_M|_{\ker F} \circ h) \\ &= f \circ (d_M|_{\ker F} \circ h + h \circ d_M|_{\ker F}) = f. \end{aligned}$$

This shows that $d \circ H + H \circ d = \text{id}_{\text{Hom}_{\hat{R}}(\ker F, M'')}$. Thus, $\text{Hom}_{\text{HMF}}(\ker F, M'') = 0$ and, therefore,

$$F^\sharp : \text{Hom}_{\hat{R}}(M', M'') \rightarrow \text{Hom}_{\hat{R}}(M, M'')$$

is a quasi-isomorphism preserving both gradings. \square

Remark 2.23. Note that $F^\sharp : \text{Hom}_{\hat{R}}(M', M'') \rightarrow \text{Hom}_{\hat{R}}(M, M'')$ maps a morphism of matrix factorizations to a morphism of matrix factorizations. By successively using this map, we can sometimes find morphisms representing a given homotopy class. This method also applies to Corollary 2.26.

The following three corollaries describe how to contract a matrix factorization using an entry in the left column. Their proofs are very close to that of propositions 2.19, 2.20 and 2.22, and are omitted.

Corollary 2.24 (strong version). *Let R be a graded commutative unital \mathbb{C} -algebra, and x a homogeneous indeterminate with $\deg x \leq 2N + 2$. Let $P : R[x] \rightarrow R$ be the evaluation map at $x = 0$, i.e. $P(f(x)) = f(0) \forall f(x) \in R[x]$.*

Suppose that $a_1, \dots, a_k, b_1, \dots, b_k$ are homogeneous elements of $R[x]$ such that

- $\deg a_j + \deg b_j = 2N + 2 \forall j = 1, \dots, k$,
- $\sum_{j=1}^k a_j b_j \in R$,
- $\exists i \in \{1, \dots, k\}$ such that $a_i = x$.

Then

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \dots & \dots \\ a_k & b_k \end{pmatrix}_{R[x]} \quad \text{and} \quad M' = \begin{pmatrix} P(a_1) & P(b_1) \\ P(a_2) & P(b_2) \\ \dots & \dots \\ P(a_{i-1}) & P(b_{i-1}) \\ P(a_{i+1}) & P(b_{i+1}) \\ \dots & \dots \\ P(a_k) & P(b_k) \end{pmatrix}_R \{q^{N+1-\deg x}\} \langle 1 \rangle$$

are homotopic as graded matrix factorizations over R .

Corollary 2.25 (weak version). *Let R be a graded commutative unital \mathbb{C} -algebra, and $a_1, \dots, a_k, b_1, \dots, b_k$ homogeneous elements of R such that $\deg a_j + \deg b_j = 2N + 2$ and $\sum_{j=1}^k a_j b_j = 0$. Then the matrix factorization*

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \dots & \dots \\ a_k & b_k \end{pmatrix}_R$$

is a chain complex. Assume that, for a given $i \in \{1, \dots, k\}$, a_i is not a zero divisor in R . Define $R' = R/(a_i)$, which inherits the grading of R . Let $P : R \rightarrow R'$ be the standard projection. Then

$$M' = \begin{pmatrix} P(a_1) & P(b_1) \\ P(a_2) & P(b_2) \\ \dots & \dots \\ P(a_{i-1}) & P(b_{i-1}) \\ P(a_{i+1}) & P(b_{i+1}) \\ \dots & \dots \\ P(a_k) & P(b_k) \end{pmatrix}_{R'}$$

is also a chain complex. And $H(M) \cong H(M')\{q^{N+1-\deg a_i}\}\langle 1 \rangle$ as graded R -modules.

Corollary 2.26 (dual version). *Let R be a graded commutative unital \mathbb{C} -algebra, and \hat{R} a graded commutative unital sub-algebra of R such that R is a free \hat{R} -module. Suppose that $a_1, \dots, a_k, b_1, \dots, b_k$ are homogeneous elements of R such that $\deg a_j + \deg b_j = 2N + 2$ and $\sum_{j=1}^k a_j b_j = w \in \hat{R}$. Assume that, for a given $i \in \{1, \dots, k\}$, a_i is not a zero divisor in R and $R' = R/(a_i)$ is also a free \hat{R} -module. Define*

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \dots & \dots \\ a_k & b_k \end{pmatrix}_R$$

and

$$M' = \begin{pmatrix} P(a_1) & P(b_1) \\ P(a_2) & P(b_2) \\ \dots & \dots \\ P(a_{i-1}) & P(b_{i-1}) \\ P(a_{i+1}) & P(b_{i+1}) \\ \dots & \dots \\ P(a_k) & P(b_k) \end{pmatrix}_{R'}$$

where $P : R \rightarrow R'$ is the standard projection. Then, for any matrix factorization M'' over \hat{R} with potential w , there is an quasi-isomorphism

$$\mathrm{Hom}_{\hat{R}}(M', M'') \rightarrow \mathrm{Hom}_{\hat{R}}(M, M'')$$

of \mathbb{Z}_2 -degree 1 and quantum degree $\deg a_i - N - 1$.

2.6. Categories of homotopically finite graded matrix factorizations. R is again a graded commutative unital \mathbb{C} -algebra in this subsection.

Definition 2.27. Let M be a graded matrix factorization over R with potential w . We say that M is homotopically finite if there exists a finitely generated graded matrix factorization \mathcal{M} over R with potential w such that $M \simeq \mathcal{M}$.

Definition 2.28. Let M and M' be any two graded matrix factorizations over R with potential w . Denote by d the differential map of $\mathrm{Hom}_R(M, M')$.

$\mathrm{Hom}_{\mathrm{MF}}(M, M')$ is defined to be the submodule of $\mathrm{Hom}_R(M, M')$ consisting of morphisms of matrix factorizations from M to M' . Or, equivalently, $\mathrm{Hom}_{\mathrm{MF}}(M, M') := \ker d$.

$\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ is defined to be the R -module of homotopy classes of morphisms of matrix factorizations from M to M' . Or, equivalently, $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ is the homology of the chain complex $(\mathrm{Hom}_R(M, M'), d)$.

It is clear that $\mathrm{Hom}_{\mathrm{MF}}(M, M')$ and $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ inherit the \mathbb{Z}_2 -grading of $\mathrm{Hom}_R(M, M')$. Recall that $\mathrm{Hom}_R(M, M')$ has a natural quantum pregrading, and d is homogeneous (with $\deg d = N + 1$.) So $\mathrm{Hom}_{\mathrm{MF}}(M, M')$ and $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ also inherit the quantum pregrading from $\mathrm{Hom}_R(M, M')$.

Definition 2.29. Let M and M' be as in Definition 2.28

$\mathrm{Hom}_{\mathrm{mf}}(M, M')$ is defined to be the \mathbb{C} -linear subspace of $\mathrm{Hom}_{\mathrm{MF}}(M, M')$ consisting of homogeneous morphisms with \mathbb{Z}_2 -degree 0 and quantum degree 0.

$\text{Hom}_{\text{hmf}}(M, M')$ is defined to be the \mathbb{C} -linear subspace of $\text{Hom}_{\text{HMF}}(M, M')$ consisting of homogeneous elements with \mathbb{Z}_2 -degree 0 and quantum degree 0.

Now we introduce four categories of homotopically finite graded matrix factorizations relevant to our construction. We require the grading of the base ring to be bounded below. We will be mainly concerned with the homotopy categories $\text{HMF}_{R,w}$ and $\text{hmf}_{R,w}$.

Definition 2.30. Let R a graded commutative unital \mathbb{C} -algebra, whose grading is bounded below. Let $w \in R$ be an homogeneous element of degree $2N + 2$. We define $\text{MF}_{R,w}$, $\text{HMF}_{R,w}$, $\text{mf}_{R,w}$ and $\text{hmf}_{R,w}$ by the following table.

| Category | Objects | Morphisms |
|--------------------|---|---------------------------|
| $\text{MF}_{R,w}$ | all homotopically finite graded matrix factorizations over R of potential w with quantum gradings bounded below | Hom_{MF} |
| $\text{HMF}_{R,w}$ | all homotopically finite graded matrix factorizations over R of potential w with quantum gradings bounded below | Hom_{HMF} |
| $\text{mf}_{R,w}$ | all homotopically finite graded matrix factorizations over R of potential w with quantum gradings bounded below | Hom_{mf} |
| $\text{hmf}_{R,w}$ | all homotopically finite graded matrix factorizations over R of potential w with quantum gradings bounded below | Hom_{hmf} |

- Remark 2.31.* (i) The above categories are additive.
(ii) The definitions of these categories here are slightly different from those in [18].
(iii) The grading of a finitely generated graded matrix factorization over R is bounded below. So finitely generated graded matrix factorizations are objects of the above categories.
(iv) Comparing Definition 2.30 to Definition 2.7, one can see that, for any object M and M' of the above categories, $M \cong M'$ means they are isomorphic as objects of $\text{mf}_{R,w}$, and $M \simeq M'$ means they are isomorphic as objects of $\text{hmf}_{R,w}$.

Lemma 2.32. *Let M and M' be any two graded matrix factorizations over R with potential w . Assume that M is homotopically finite. Then the quantum pregrading on $\text{Hom}_{\text{HMF}}(M, M')$ is a grading.*

In particular, if the grading of R is bounded below and M and M' are objects of $\text{MF}_{R,w}$, then $\text{Hom}_{\text{HMF}}(M, M')$ has a quantum grading.

Proof. Since M is homotopically finite, there is a finitely generated graded matrix factorization \mathcal{M} over R with potential w such that $M \simeq \mathcal{M}$. That is, there exist homogeneous morphisms $f : M \rightarrow \mathcal{M}$ and $g : \mathcal{M} \rightarrow M$ preserving both the \mathbb{Z}_2 -grading and the quantum pregrading such that $g \circ f \simeq \text{id}_M$ and $f \circ g \simeq \text{id}_{\mathcal{M}}$.

Denote by $d_M, d_{M'}, d$ the differential maps of M, M' and $\text{Hom}_R(M, M')$. Let $f^\# : \text{Hom}_R(\mathcal{M}, M') \rightarrow \text{Hom}_R(M, M')$ and $g^\# : \text{Hom}_R(M, M') \rightarrow \text{Hom}_R(\mathcal{M}, M')$ be the R -module maps induced by f and g . One can easily check that $f^\#$ and $g^\#$ are chain maps. Since $g \circ f \simeq \text{id}_M$, we know that there exist an homogeneous R -module map $h : M \rightarrow M$ of \mathbb{Z}_2 -degree 1 and quantum degree $-N - 1$ such that

$$g \circ f - \text{id}_M = d_M \circ h + h \circ d_M.$$

Define an R -module map $H : \text{Hom}_R(M, M') \rightarrow \text{Hom}_R(M, M')$ so that, for any $\alpha \in \text{Hom}_R(M, M')$ with \mathbb{Z}_2 -degree ε , $H(\alpha) = (-1)^\varepsilon \alpha \circ h$. Then, for such an α , we

have

$$\begin{aligned}
& (dH + Hd)(\alpha) \\
&= (-1)^\varepsilon d(\alpha \circ h) + (-1)^{\varepsilon+1} (d\alpha) \circ h \\
&= (-1)^\varepsilon (d_{M'} \circ \alpha \circ h - (-1)^{\varepsilon+1} \alpha \circ h \circ d_M) + (-1)^{\varepsilon+1} (d_{M'} \circ \alpha - (-1)^\varepsilon \alpha \circ d_M) \circ h \\
&= \alpha \circ (h \circ d_M + d_M \circ h) = \alpha \circ (g \circ f - \text{id}_M) \\
&= f^\sharp \circ g^\sharp(\alpha) - \alpha.
\end{aligned}$$

This shows that $f^\sharp \circ g^\sharp \simeq \text{id}_{\text{Hom}_R(M, M')}$. Similarly, $g^\sharp \circ f^\sharp \simeq \text{id}_{\text{Hom}_R(\mathcal{M}, M')}$. Thus, $\text{Hom}_R(M, M') \simeq \text{Hom}_R(\mathcal{M}, M')$ and this homotopy equivalence preserves both the \mathbb{Z}_2 -grading and the quantum pregrading. So $\text{Hom}_{\text{HMF}}(M, M') \cong \text{Hom}_{\text{HMF}}(\mathcal{M}, M')$ and the isomorphism preserves both the \mathbb{Z}_2 -grading and the quantum pregrading. But, by Lemma 2.6, the quantum pregrading of $\text{Hom}_R(\mathcal{M}, M')$ is a grading. So the quantum pregrading of $\text{Hom}_{\text{HMF}}(M, M') \cong \text{Hom}_{\text{HMF}}(\mathcal{M}, M')$ is also a grading. \square

2.7. Categories of chain complexes. Now we introduce our notations for categories of chain complexes.

Definition 2.33. Let \mathcal{C} be an additive category. We denote by $\text{Ch}^b(\mathcal{C})$ the category of bounded chain complexes over \mathcal{C} . More precisely,

- An object of $\text{Ch}^b(\mathcal{C})$ is a chain complex

$$(2.3) \quad \cdots \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} A_{i+2} \xrightarrow{d_{i+2}} \cdots$$

where A_i 's are objects of \mathcal{C} , d_i 's are morphisms of \mathcal{C} such that $d_{i+1} \circ d_i = 0$ for $i \in \mathbb{Z}$, and there exists integers $k \leq K$ such that $A_i = 0$ if $i > K$ or $i < k$.

- A morphism f of $\text{Ch}^b(\mathcal{C})$ is a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_{i-1}} & A_i & \xrightarrow{d_i} & A_{i+1} & \xrightarrow{d_{i+1}} & A_{i+2} \xrightarrow{d_{i+2}} \cdots, \\
& & \downarrow f_i & & \downarrow f_{i+1} & & \downarrow f_{i+2} \\
\cdots & \xrightarrow{d'_{i-1}} & A'_i & \xrightarrow{d'_i} & A'_{i+1} & \xrightarrow{d'_{i+1}} & A'_{i+2} \xrightarrow{d'_{i+2}} \cdots
\end{array}$$

where each row is an object of $\text{Ch}^b(\mathcal{C})$ and vertical arrows are morphisms of \mathcal{C} .

Chain homotopy in $\text{Ch}^b(\mathcal{C})$ is defined the usual way.

We denote by $\text{hCh}^b(\mathcal{C})$ the homotopy category of chain complexes over \mathcal{C} , or simply the homotopy category of \mathcal{C} . $\text{hCh}^b(\mathcal{C})$ is defined by

- An object of $\text{hCh}^b(\mathcal{C})$ is an object of $\text{Ch}^b(\mathcal{C})$.
- For any two objects A and B of $\text{hCh}^b(\mathcal{C})$ $\text{Hom}_{\text{hCh}^b(\mathcal{C})}(A, B)$ is $\text{Hom}_{\text{Ch}^b(\mathcal{C})}(A, B)$ modulo the subgroup of null homotopic morphisms.

An isomorphism in $\text{Ch}^b(\mathcal{C})$ is denoted by “ \cong ”. An isomorphism in $\text{hCh}^b(\mathcal{C})$ is commonly known as a homotopy equivalence and denoted by “ \simeq ”.

Let A be the object of $\text{Ch}^b(\mathcal{C})$ (and $\text{hCh}^b(\mathcal{C})$) given in (2.3). Then A admits an obvious bounded homological grading \deg_h with $\deg_h A_i = i$. Morphisms of $\text{Ch}^b(\mathcal{C})$ and $\text{hCh}^b(\mathcal{C})$ preserve this grading. Denote by $A\|k\|$ the object of $\text{Ch}^b(\mathcal{C})$ obtained by shifting the homological grading by k . That is, $A\|k\|$ is the same chain complex as A except that $\deg_h A_i = i + k$ in $A\|k\|$.

Let us try to understand how to compute $\mathrm{Hom}_{\mathrm{Ch}^b(\mathcal{C})}(A, B)$ and $\mathrm{Hom}_{\mathrm{hCh}^b(\mathcal{C})}(A, B)$ for objects A, B of $\mathrm{Ch}^b(\mathcal{C})$.

Definition 2.34. Let \mathcal{C} be an additive category, and $(A, d), (B, d')$ objects of $\mathrm{Ch}^b(\mathcal{C})$. Let $\mathrm{Kom}^0(A, B)$ be the set of diagrams of the form

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{i-1}} & A_i & \xrightarrow{d_i} & A_{i+1} & \xrightarrow{d_{i+1}} & A_{i+2} \xrightarrow{d_{i+2}} \cdots, \\ & & \downarrow f_i & & \downarrow f_{i+1} & & \downarrow f_{i+2} \\ \cdots & \xrightarrow{d'_{i-1}} & B_i & \xrightarrow{d'_i} & B_{i+1} & \xrightarrow{d'_{i+1}} & B_{i+2} \xrightarrow{d'_{i+2}} \cdots \end{array}$$

where vertical arrows are morphisms of \mathcal{C} , and we do not require any commutativity. Note that $\mathrm{Kom}^0(A, B)$ is an abelian group.

For any $k \in \mathbb{Z}$, define $\mathrm{Kom}^k(A, B) := \mathrm{Kom}^0(A[k], B)$. Note that, if $f \in \mathrm{Kom}^k(A, B)$, then $D_k f := f \circ d - (-1)^k d' \circ f$ is an element of $\mathrm{Kom}^{k+1}(A, B)$. Clearly,

$$(\mathrm{Kom}(A, B) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Kom}^k(A, B), \quad D := \bigoplus_{k \in \mathbb{Z}} D_k)$$

is a bounded chain complex of abelian groups with an obvious homological grading, in which $\mathrm{Kom}^k(A, B)$ has grading k .

The following lemma is obvious from the definitions of $\mathrm{Hom}_{\mathrm{Ch}^b(\mathcal{C})}(A, B)$ and $\mathrm{Hom}_{\mathrm{hCh}^b(\mathcal{C})}(A, B)$.

Lemma 2.35. *Using notations from Definition 2.34, we have*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ch}^b(\mathcal{C})}(A, B) &= \ker D_0, \\ \mathrm{Hom}_{\mathrm{hCh}^b(\mathcal{C})}(A, B) &= H^0(\mathrm{Kom}(A, B), D). \end{aligned}$$

3. GRADED MATRIX FACTORIZATIONS OVER A POLYNOMIAL RING

In this section, we review the algebraic properties of graded matrix factorizations over polynomial rings. Most of these properties can be found in [18].

In the rest of this section, we assume $R = \mathbb{C}[X_1, \dots, X_m]$ is a polynomial ring over \mathbb{C} , where X_1, \dots, X_m are homogeneous indeterminants of positive integer degrees. There is a natural grading $\{R^{(i)}\}$ of R . It is clear that, for each i , $R^{(i)}$ is finite dimensional. In particular, $R^{(i)} = 0$ if $i < 0$ and $R^{(0)} = \mathbb{C}$. Also, R has a unique maximal homogeneous ideal $\mathfrak{J} = (X_1, \dots, X_m)$.

For a homogeneous element $w \in \mathfrak{J}$ of degree $2N + 2$, the Jacobian ideal of w is defined to be $I_w = (\frac{\partial w}{\partial X_1}, \dots, \frac{\partial w}{\partial X_m})$. We call w non-degenerate if the Jacobian algebra $R_w := R/I_w$ is finite dimensional over \mathbb{C} . Otherwise, we call w degenerate. Note that, for any homogeneous element $w \in \mathfrak{J}$ of degree $2N + 2$, Euler's formula gives that

$$w = \frac{1}{2N + 2} \sum_{i=1}^m (\deg X_i) \cdot X_i \frac{\partial w}{\partial X_i}.$$

Thus, w is in its Jacobian ideal.

Lemma 3.1. [18, Propositions 5] *Let M and M' be objects of $\mathrm{HMF}_{R,w}$. Then the action of R on $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ factors through the Jacobian ring R_w .*

Proof. (Following [18].) Choose a basis for M and express the differential d of M as a matrix D . Differentiating $D^2 = w \cdot \text{id}$ by X_i , we get $\frac{\partial D}{\partial X_i} \circ D + D \circ \frac{\partial D}{\partial X_i} = \frac{\partial w}{\partial X_i} \cdot \text{id}$. So multiplication by $\frac{\partial w}{\partial X_i}$ on M is a morphism homotopic to 0. Thus multiplication by $\frac{\partial w}{\partial X_i}$ on $\text{Hom}_{\text{HMF}}(M, M')$ is the zero map. \square

3.1. Homogeneous basis. In general, a free graded module over a graded ring is not necessarily graded-free, i.e. need not have a basis consisting of homogeneous elements. (c.f. Definition 2.2.) However, if the base ring is R , and the grading on the free module is bounded below, then the module has a homogeneous basis. We prove this using argument in [30, Chapter 13]. First, we introduce the following definition from [30, Chapter 13].

Definition 3.2. Let P be a graded R -module. We say that P is graded projective if and only if, whenever we have a diagram

$$\begin{array}{ccc} & P & \\ & \downarrow \beta & \\ V & \xrightarrow{\alpha} W & \longrightarrow 0 \end{array}$$

of graded R -modules with exact row, where α and β are homogeneous R -module maps preserving the grading, there exists a homogeneous R -module map $\gamma : P \rightarrow V$ that preserves the grading and makes the following diagram commutative.

$$\begin{array}{ccc} & P & \\ \swarrow \gamma & \downarrow \beta & \\ V & \xrightarrow{\alpha} W & \longrightarrow 0 \end{array}$$

Lemma 3.3. [30] *Let M be a free graded R -module whose grading is bounded below. Then M is graded-free over R , i.e. M has a homogeneous basis over R .*

In particular, for any homogeneous element $w \in \mathfrak{I}$ of degree $2N+2$, every object of $\text{MF}_{R,w}$ has a homogeneous basis.

Proof. Since M is graded and free, it is graded and projective. By [30, Lemma 13.3], M is graded-projective. Recall that $R^{(0)} = \mathbb{C}$ and any \mathbb{C} -linear space has a basis. So, according to [30, Exercise 3, page 130], M is graded-free.

By definition, the quantum grading of every object of $\text{MF}_{R,w}$ is bounded below. So the above argument applies to objects of $\text{MF}_{R,w}$. \square

3.2. Homology of graded matrix factorizations over R . Let $w \in \mathfrak{I}$ be a homogeneous element of degree $2N+2$, and M a graded matrix factorization over R with potential w . Note that $M/\mathfrak{I}M$ is a chain complex over \mathbb{C} , and it inherits the gradings of M .

Definition 3.4. $H_R(M)$ is defined to be the homology of $M/\mathfrak{I}M$. It inherits the gradings of M . If R is clear from the context, we drop it from the notations.

Denote by $H_R^{j,\varepsilon}(M)$ the subspace of $H_R(M)$ consisting of homogeneous elements of quantum degree j and \mathbb{Z}_2 -degree ε . Following [18], we define the graded dimension of M to be

$$\text{gdim}_R(M) = \sum_{j,\varepsilon} q^j \tau^\varepsilon \dim_{\mathbb{C}} H_R^{j,\varepsilon}(M) \in \mathbb{Z}[[q]][\tau]/(\tau^2 - 1).$$

Again, if R is clear from the context, we drop it from the notations.

Remark 3.5. One needs to be careful when dropping R from the notations. For example, when $w = 0$, M is itself a chain complex. Denote by $H_{\mathbb{C}}(M)$ the usual homology of M . Then, in general, $H_R(M) \neq H_{\mathbb{C}}(M)$. So carelessly dropping R from the notations in this case may lead to confusion.

Any homogeneous morphism of graded matrix factorizations induces a homogeneous homomorphism of the homology, and homotopic morphisms induce the same homomorphism of the homology. In particular, $f : M \xrightarrow{\sim} M'$ is a homotopy equivalence implies that it induces map $f_* : H_R(M) \xrightarrow{\cong} H_R(M')$ is an isomorphism. Surprisingly, according to [18, Proposition 8], the converse is also true. Next we review properties of the homology of matrix factorizations given in [18].

Lemma 3.6. *Let M be a free graded R -module, whose grading is bounded below. Let $V = M/\mathfrak{I}M$. Then there is a homogeneous R -module map $F : V \otimes_{\mathbb{C}} R \rightarrow M$ preserving the grading. In particular, if $\{v_{\beta} | \beta \in \mathcal{B}\}$ is a homogeneous \mathbb{C} -basis for V , then $\{F(v_{\beta} \otimes 1) | \beta \in \mathcal{B}\}$ is a homogeneous R -basis for M .*

Proof. By Lemma 3.3, M has a homogeneous basis $\{e_{\alpha} | \alpha \in \mathcal{A}\}$. Then, as graded vector spaces, $V \cong \bigoplus_{\alpha \in \mathcal{A}} \mathbb{C} \cdot e_{\alpha}$. So, as graded R -modules,

$$M \cong \bigoplus_{\alpha \in \mathcal{A}} R \cdot e_{\alpha} \cong V \otimes_{\mathbb{C}} R.$$

This proves the existence of F . The second part of the lemma follows easily. \square

The next proposition is a reformulation of [18, Proposition 7]. For the convenience of the reader, we give a detailed proof.

Proposition 3.7. [18, Proposition 7] *Let M be a graded matrix factorization over R with homogeneous potential $w \in \mathfrak{I}$ of degree $2N+2$. Assume the quantum grading of M is bounded below. Then there exist graded matrix factorizations M_c and M_{es} over R with potential w such that*

- (i) $M \cong M_c \oplus M_{es}$,
- (ii) $M_c \simeq 0$ and, therefore, $M \simeq M_{es}$,
- (iii) $M_{es} \cong H_R(M) \otimes_{\mathbb{C}} R$ as graded R -modules, and $H_R(M) \cong M_{es}/\mathfrak{I}M_{es}$ as graded \mathbb{C} -spaces.

Proof. (Following [18].) Write M as $M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0$. Then the chain complex $V := M/\mathfrak{I}M$ is given by $V_0 \xrightarrow{\hat{d}_0} V_1 \xrightarrow{\hat{d}_1} V_0$, where $V_{\varepsilon} = M_{\varepsilon}/\mathfrak{I}M_{\varepsilon}$ for $\varepsilon = 0, 1$. By Lemma 3.3, M_{ε} has a homogeneous basis $\{e_{\sigma} | \sigma \in \mathcal{S}_{\varepsilon}\}$, which induces a homogeneous \mathbb{C} -basis $\{\hat{e}_{\sigma} | \sigma \in \mathcal{S}_{\varepsilon}\}$ for V_{ε} . Under this homogeneous basis, the entries of matrices of d_0 and d_1 are homogeneous polynomials. And the matrices of \hat{d}_0 and \hat{d}_1 are obtained by letting $X_1 = \dots = X_m = 0$ in the matrices of d_0 and d_1 , which keep the scalar entries and kills entries with positive degrees.

We call $\{(\hat{u}_{\rho}, \hat{v}_{\rho}) | \rho \in \mathcal{P}\}$ a “good” set if

- $\{\hat{u}_{\rho} | \rho \in \mathcal{P}\}$ is set of linearly independent homogeneous elements in V_0 ,
- $\{\hat{v}_{\rho} | \rho \in \mathcal{P}\}$ is set of linearly independent homogeneous elements in V_1 ,
- $\hat{d}_0(\hat{u}_{\rho}) = \hat{v}_{\rho}$ and $\hat{d}_1(\hat{v}_{\rho}) = 0$.

Using Zorn’s Lemma, we find a maximal “good” set $G = \{(\hat{u}_{\alpha}, \hat{v}_{\alpha}) | \alpha \in \mathcal{A}\}$. Using Zorn’s Lemma again, we extend $\{\hat{u}_{\alpha} | \alpha \in \mathcal{A}\}$ into a homogeneous basis $\{\hat{u}_{\alpha} | \alpha \in \mathcal{A} \cup \mathcal{B}_0\}$ for V_0 , and $\{\hat{v}_{\alpha} | \alpha \in \mathcal{A}\}$ into a homogeneous basis $\{\hat{v}_{\alpha} | \alpha \in \mathcal{A} \cup \mathcal{B}_1\}$ for V_1 .

For each $\beta \in \mathcal{B}_0$, we can write $\hat{d}_0 \hat{u}_\beta = \sum_{\alpha \in \mathcal{A} \cup \mathcal{B}_1} c_{\alpha\beta} \cdot \hat{v}_\alpha$, where $c_{\alpha\beta} \in \mathbb{C}$, and the right hand side is a finite sum.

By Lemma 3.6, there is a homogeneous isomorphism $F_\varepsilon : V_\varepsilon \otimes_{\mathbb{C}} R \xrightarrow{\cong} M_\varepsilon$ preserving both gradings. Let $u_\alpha = F_0(\hat{u}_\alpha \otimes 1)$ and $v_\alpha = F_1(\hat{v}_\alpha \otimes 1)$. Then $\{u_\alpha | \alpha \in \mathcal{A} \cup \mathcal{B}_0\}$ and $\{v_\alpha | \alpha \in \mathcal{A} \cup \mathcal{B}_1\}$ are homogeneous R -bases for M_0 and M_1 . Under this basis, we have that

$$du_\alpha = v_\alpha + \sum_{\beta \in \mathcal{A} \cup \mathcal{B}_1, \beta \neq \alpha} f_{\beta\alpha} v_\beta,$$

where $f_{\beta\alpha} \in \mathfrak{J}$ and the sum on the right hand side is a finite sum. That is, for each α ,

$$(3.1) \quad f_{\beta\alpha} = 0 \text{ for all but finitely many } \beta.$$

Also, using that $\hat{d}_0(\hat{u}_\alpha) = \hat{v}_\alpha$ for $\alpha \in \mathcal{A}$, one can see that

$$(3.2) \quad f_{\beta\alpha} = 0 \text{ if } \beta \neq \alpha \text{ and } \deg v_\beta \geq \deg v_\alpha.$$

For $\alpha \in \mathcal{A}$ and $k > 0$, let

$$C_{*\alpha}^k = \{(\gamma_0, \dots, \gamma_k) \in \mathcal{A}^{k+1} | \gamma_k = \alpha, \deg v_{\gamma_0} < \dots < \deg v_{\gamma_k}, f_{\gamma_0\gamma_1} \cdots f_{\gamma_{k-1}\gamma_k} \neq 0\}.$$

By (3.1), $C_{*\alpha}^k$ is a finite set. For each α , $C_{*\alpha}^k = \emptyset$ for large k 's since the quantum grading of M is bounded below. For $\alpha, \beta \in \mathcal{A}$ and $k > 0$, let

$$C_{\beta\alpha}^k = \{(\gamma_0, \dots, \gamma_k) \in C_{*\alpha}^k | \gamma_0 = \beta\}.$$

Then $\cup_{\beta \in \mathcal{A}} C_{\beta\alpha}^k = C_{*\alpha}^k$. So each $C_{\beta\alpha}^k$ is finite. And, for each k , $C_{\beta\alpha}^k \neq \emptyset$ for only finitely many β . Also, by definition, it is easy to see that $C_{\beta\alpha}^k = \emptyset$ if $\deg v_\beta \geq \deg v_\alpha$. Moreover, for each α , there is a $k_0 > 0$ such that $C_{\beta\alpha}^k = \emptyset$ for any β whenever $k > k_0$.

Now define $t_{\beta\alpha} \in R$ by

$$t_{\beta\alpha} = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha, \deg v_\beta \geq \deg v_\alpha, \\ \sum_{k \geq 1} (-1)^k \sum_{(\gamma_0, \dots, \gamma_k) \in C_{\beta\alpha}^k} f_{\gamma_0\gamma_1} \cdots f_{\gamma_{k-1}\gamma_k} & \text{if } \deg v_\beta < \deg v_\alpha. \end{cases}$$

From the above discussion, we know that the sum on the right hand side is always finite. So $t_{\beta\alpha}$ is well defined. Furthermore, given an $\alpha \in \mathcal{A}$, $t_{\beta\alpha} = 0$ for all but finitely many β . So, for $\alpha \in \mathcal{A}$, $u'_\alpha := \sum_{\beta \in \mathcal{A}} t_{\beta\alpha} u_\beta$ is well defined. And $\{u'_\alpha | \alpha \in \mathcal{A}\} \cup \{u_\beta | \beta \in \mathcal{B}_0\}$ is also a homogeneous R -basis for M_0 . One can check that, for $\alpha \in \mathcal{A}$,

$$du'_\alpha = v_\alpha + \sum_{\beta \in \mathcal{B}_1} f'_{\beta\alpha} v_\beta,$$

where the right hand side is a finite sum, and $f'_{\beta\alpha} \in \mathfrak{J}$. Now let

$$v'_\alpha = \begin{cases} v_\alpha + \sum_{\beta \in \mathcal{B}_1} f'_{\beta\alpha} v_\beta & \text{if } \alpha \in \mathcal{A}, \\ v_\alpha & \text{if } \alpha \in \mathcal{B}_1. \end{cases}$$

Then $\{v'_\alpha | \alpha \in \mathcal{A} \cup \mathcal{B}_1\}$ is a homogeneous R -basis for M_1 . Under this basis, we have

$$\begin{cases} du'_\alpha = v'_\alpha & \text{if } \alpha \in \mathcal{A}, \\ du_\beta = \sum_{\alpha \in \mathcal{A}} g_{\alpha\beta} v'_\alpha + \sum_{\gamma \in \mathcal{B}_1} g_{\gamma\beta} v'_\gamma & \text{if } \beta \in \mathcal{B}_0, \end{cases}$$

where the sums on the right hand side are finite. For $\beta \in \mathcal{B}_0$, we let

$$u'_\beta = u_\beta - \sum_{\alpha \in \mathcal{A}} g_{\alpha\beta} u'_\alpha.$$

Then $\{u'_\alpha | \alpha \in \mathcal{A} \cup \mathcal{B}_0\}$ is again a homogeneous R -basis for M_0 , and

$$\begin{cases} du'_\alpha = v'_\alpha & \text{if } \alpha \in \mathcal{A}, \\ du'_\beta = \sum_{\gamma \in \mathcal{B}_1} g'_{\gamma\beta} v'_\gamma & \text{if } \beta \in \mathcal{B}_0, \end{cases}$$

where the sum on the right hand side is finite. Using that $d^2 = \text{wid}_M$, one can check that

$$\begin{cases} dv'_\alpha = w \cdot v'_\alpha & \text{if } \alpha \in \mathcal{A}, \\ dv'_\beta = \sum_{\gamma \in \mathcal{B}_0} g''_{\gamma\beta} u'_\gamma & \text{if } \beta \in \mathcal{B}_1, \end{cases}$$

where the sum on the right hand side is finite.

Define $M_{(1,w)}$ to be the submodule of M spanned by $\{u'_\alpha | \alpha \in \mathcal{A}\} \cup \{v'_\alpha | \alpha \in \mathcal{A}\}$, and M' the submodule of M spanned by $\{u'_\beta | \beta \in \mathcal{B}_0\} \cup \{v'_\beta | \beta \in \mathcal{B}_1\}$. Then $M_{(1,w)}$ and M' are both graded matrix factorizations and $M = M_{(1,w)} \oplus M'$. Note that

- (a) $M_{(1,w)}$ is a direct sum of components of the form $(1,w)_R\{q^k\}$,
- (b) Under the standard projection $M \rightarrow M/\mathfrak{I}M$, we have, for $\alpha \in \mathcal{A}$, $u'_\alpha \mapsto \hat{u}_\alpha$ and $v'_\alpha \mapsto \hat{v}_\alpha$.

In particular, (b) above means that M' does not have direct sum components of the form $(1,w)_R\{q^k\}$. Otherwise, we can enlarge the “good” set G , which contradicts the fact that G is maximal. We then apply a similar argument to M' and find a decomposition $M' = M_{(w,1)} \oplus M_{es}$ of graded matrix factorizations satisfying

- $M_{(w,1)}$ is a direct sum of components of the form $(w,1)_R\{q^k\}$,
- M_{es} has no direct sum component of the form $(1,w)_R\{q^k\}$ or $(w,1)_R\{q^k\}$.

Let $M_c = M_{(1,w)} \oplus M_{(w,1)}$. Then $M = M_c \oplus M_{es}$. Since $(1,w)_R\{q^k\}$ and $(w,1)_R\{q^k\}$ are both homotopic to 0, $M_c \simeq 0$. So $M \simeq M_{es}$. It is clear that, under any homogeneous basis for M_{es} , all entries of the matrices representing the differential map of M_{es} must be in \mathfrak{I} . Otherwise, a simple change of basis would show that M_{es} has a component of the form $(1,w)_R\{q^k\}$ or $(w,1)_R\{q^k\}$. Therefore, $H_R(M) \cong H_R(M_{es}) \cong M_{es}/\mathfrak{I}M_{es}$. So, by Lemma 3.6, $M_{es} \cong H_R(M) \otimes_{\mathbb{C}} R$ as graded modules. \square

The following corollaries are from [18].

Corollary 3.8. [18, Proposition 8] *Let M and M' be graded matrix factorizations over R with homogeneous potential $w \in \mathfrak{I}$ of degree $2N + 2$. Assume the quantum gradings of M and M' are bounded below. Suppose that $f : M \rightarrow M'$ is a homogeneous morphism preserving both gradings. Then f is a homotopy equivalence if and only if it induces an isomorphism of the homology $f_* : H_R(M) \rightarrow H_R(M')$.*

Proof. (Following [18].) If f is a homotopy equivalence, then f_* is clearly an isomorphism. Let us now prove the converse. Assume f_* is an isomorphism. Let $M = M_c \oplus M_{es}$ and $M' = M'_c \oplus M'_{es}$ be decompositions of M and M' given by Proposition 3.7. So f induces a morphism $f_{es} : M_{es} \rightarrow M'_{es}$. Note that $H_R(M) \cong M_{es}/\mathfrak{I}M_{es}$, $H_R(M') \cong M'_{es}/\mathfrak{I}M'_{es}$, $M_{es} = H_R(M) \otimes_{\mathbb{C}} R$ and $M'_{es} = H_R(M') \otimes_{\mathbb{C}} R$. So f_{es} is an isomorphism since f_* is an isomorphism. It follows that f is a homotopy equivalence. \square

Corollary 3.9. [18, Proposition 7] *Let M be a graded matrix factorization over R with homogeneous potential $w \in \mathfrak{J}$ of degree $2N + 2$. Assume the quantum grading of M is bounded below. Then*

- (i) $M \simeq 0$ if and only if $H_R(M) = 0$ or, equivalently, $\mathrm{gdim}_R(M) = 0$;
- (ii) M is homotopically finite if and only if $H_R(M)$ is finite dimensional over \mathbb{C} or, equivalently, $\mathrm{gdim}_R(M) \in \mathbb{Z}[q, \tau]/(\tau^2 - 1)$.

Proof. For (i), we have

$$M \simeq 0 \Rightarrow H_R(M) = 0 \Rightarrow M_{es} \cong H_R(M) \otimes_{\mathbb{C}} R = 0 \Rightarrow M \simeq 0.$$

Now consider (ii). If M is homotopically finite, then there is a finitely generated graded matrix factorization \mathcal{M} such that $M \simeq \mathcal{M}$. Note that $\mathcal{M}/\mathfrak{J}\mathcal{M}$ is finite dimensional over \mathbb{C} . This implies that $H_R(M) \cong H_R(\mathcal{M})$ is finite dimensional over \mathbb{C} . If $H_R(M)$ is finite dimensional over \mathbb{C} , then $M_{es} \cong H_R(M) \otimes_{\mathbb{C}} R$ is finitely generated over R . But $M \simeq M_{es}$. So M is homotopically finite. \square

3.3. The Krull-Schmidt property. In this subsection, we review the Krull-Schmidt property of matrix factorizations and chain complexes of matrix factorizations. We follow the approach in [8, Section 1] and [18, Section 5].

Definition 3.10. [8] An additive category \mathcal{C} is called a \mathbb{C} -category if all morphism sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are \mathbb{C} -linear spaces and the composition of morphisms is \mathbb{C} -bilinear.

A \mathbb{C} -category \mathcal{C} is called fully additive if any idempotent morphism of \mathcal{C} splits, i.e. defines a decomposition into a direct sum.

A \mathbb{C} -category \mathcal{C} is called locally finite dimensional if, for every pair A, B of objects of \mathcal{C} , $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is finite dimensional over \mathbb{C} .

A \mathbb{C} -category \mathcal{C} is called Krull-Schmidt if

- every object of \mathcal{C} is isomorphic to a finite direct sum $A_1 \oplus \cdots \oplus A_n$ of indecomposable objects of \mathcal{C} ;
- and, if $A_1 \oplus \cdots \oplus A_n \cong A'_1 \oplus \cdots \oplus A'_l$, where $A_1, \dots, A_n, A'_1, \dots, A'_l$ are indecomposable objects of \mathcal{C} , then $n = l$ and there is a permutation σ of $\{1, \dots, n\}$ such that $A_i \cong A'_{\sigma(i)}$ for $i = 1, \dots, n$.

Note that, for any homogeneous $w \in \mathfrak{J}$ of degree $2N + 2$, the categories $\mathrm{MF}_{R,w}$, $\mathrm{HMF}_{R,w}$, $\mathrm{mf}_{R,w}$ and $\mathrm{hmf}_{R,w}$ are all \mathbb{C} -categories. Moreover, if \mathcal{C} is a \mathbb{C} -category, then $\mathrm{Ch}^b(\mathcal{C})$ and $\mathrm{hCh}^b(\mathcal{C})$ are both \mathbb{C} -categories. Then following lemma is from [8, Section 1].

Lemma 3.11. [8, Section 1] *If \mathcal{C} is a fully additive and locally finite dimensional \mathbb{C} -category, then \mathcal{C} is Krull-Schmidt.*

Moreover, if \mathcal{C} is a fully additive and locally finite dimensional \mathbb{C} -category, then $\mathrm{Ch}^b(\mathcal{C})$ and $\mathrm{hCh}^b(\mathcal{C})$ are both fully additive, locally finite dimensional and, therefore, Krull-Schmidt.

Sketch of proof. (Following [8].) A \mathbb{C} -category \mathcal{C} is called local if every object of \mathcal{C} decomposes into a finite direct sum of objects with local endomorphism rings. One can check that \mathcal{C} is local if it is fully additive and locally finite dimensional. By [2, Theorem 3.6], local \mathbb{C} -categories are Krull-Schmidt. So fully additive locally finite dimensional \mathbb{C} -categories are Krull-Schmidt.

If \mathcal{C} is a fully additive and locally finite dimensional \mathbb{C} -category, then $\mathrm{Ch}^b(\mathcal{C})$ is also fully additive and locally finite dimensional. So $\mathrm{Ch}^b(\mathcal{C})$ is local and, therefore,

Krull-Schmidt. For every pair (A, B) of objects of $\mathbf{hCh}^b(\mathcal{C})$, $\mathrm{Hom}_{\mathbf{hCh}^b(\mathcal{C})}(A, B)$ is a quotient space of $\mathrm{Hom}_{\mathbf{Ch}^b(\mathcal{C})}(A, B)$. Thus, $\mathbf{hCh}^b(\mathcal{C})$ is also locally finite dimensional. Since $\mathbf{Ch}^b(\mathcal{C})$ is local, any object A of $\mathbf{hCh}^b(\mathcal{C})$ decomposes into

$$A \cong A_1 \oplus \cdots \oplus A_m,$$

where $\mathrm{Hom}_{\mathbf{Ch}^b(\mathcal{C})}(A_i, A_i)$ is a local ring for each $i = 1, \dots, m$. But $\mathrm{Hom}_{\mathbf{hCh}^b(\mathcal{C})}(A_i, A_i)$ is a quotient ring of $\mathrm{Hom}_{\mathbf{Ch}^b(\mathcal{C})}(A_i, A_i)$. So, for each i , $\mathrm{Hom}_{\mathbf{hCh}^b(\mathcal{C})}(A_i, A_i)$ is either a local ring or 0. In the latter case, A_i is homotopic to 0. This shows that $\mathbf{hCh}^b(\mathcal{C})$ is local and, therefore, Krull-Schmidt. Since local \mathbb{C} -categories are fully additive, $\mathbf{hCh}^b(\mathcal{C})$ is also fully additive. \square

Remark 3.12. In [8, Section 1], the above lemma is actually proved for categories over any complete local Noetherian ring. (It is trivial to verify that \mathbb{C} is a complete local Noetherian ring.)

In the rest of this subsection, we assume that w is a homogeneous element of \mathfrak{J} with $\deg w = 2N + 2$. The next lemma is the lifting idempotent property from [18, Section 5].

Lemma 3.13. [18, Section 5] *Let M be a finitely generated graded matrix factorization over R with potential w . If a homogeneous morphism $f : M \rightarrow M$ of matrix factorizations preserves both gradings of M and satisfies $f \circ f \simeq f$, then there is a homogeneous morphism $g : M \rightarrow M$ of matrix factorizations preserving both gradings of M that satisfies $g \simeq f$ and $g \circ g = g$.*

Proof. (Following [18].) Let $P : M \rightarrow M_{es}$ and $J : M_{es} \rightarrow M$ be the projection and inclusion from the decomposition in Proposition 3.7. Then f induces a morphism $f_{es} = P \circ f \circ J : M_{es} \rightarrow M_{es}$, which satisfies $f_{es} \circ f_{es} \simeq f_{es}$.

Let

$$\alpha : \mathrm{Hom}_{\mathrm{mf}}(M_{es}, M_{es}) \rightarrow \mathrm{Hom}_{\mathrm{hmf}}(M_{es}, M_{es})$$

be the natural projection taking each morphism to its homotopy class, and

$$\beta : \mathrm{Hom}_{\mathrm{mf}}(M_{es}, M_{es}) \rightarrow \mathrm{Hom}_{\mathbb{C}}(H_R(M), H_R(M))$$

the map taking each morphism to the induced map on the homology. Then $\ker \alpha$ and $\ker \beta$ are ideals of the ring $\mathrm{Hom}_{\mathrm{mf}}(M_{es}, M_{es})$, and $\ker \alpha \subset \ker \beta$.

Choose a homogeneous basis $\{e_1, \dots, e_n\}$ for M_{es} . For any $h \in \ker \beta$, let H be its matrix under this basis. Recall that $H_R(M) \cong M_{es}/\mathfrak{J}M_{es}$. Since $\beta(h) = 0$, we know that entries of H are elements of \mathfrak{J} . This implies that, if $h \in (\ker \beta)^k$, then entries of H are elements of \mathfrak{J}^k . But the matrix of a homogeneous morphism preserving the quantum grading can not have entries of arbitrarily large degrees. Thus, $(\ker \beta)^k = 0$ for $k \gg 0$ and, therefore, $(\ker \alpha)^k = 0$ for $k \gg 0$. This shows that $\ker \alpha$ is a nilpotent ideal of $\mathrm{Hom}_{\mathrm{mf}}(M_{es}, M_{es})$. By [4, Theorem 1.7.3], nilpotent ideals have the lifting idempotents property. Thus, there is a homogeneous morphism $g_{es} : M_{es} \rightarrow M_{es}$ of matrix factorizations preserving both gradings of M_{es} that satisfies $g_{es} \simeq f_{es}$ and $g_{es} \circ g_{es} = g_{es}$.

Now define a morphism $g : M \rightarrow M$ by $g = J \circ g_{es} \circ P$. It is easy to check that g preserves both gradings of M and satisfies $g \simeq f$ and $g \circ g = g$. \square

Lemma 3.14. [18, Proposition 24] *$\mathrm{hmf}_{R,w}$ is fully additive.*

Proof. (Following [18].) Let M be an object of $\mathrm{hmf}_{R,w}$, and $f : M \rightarrow M$ a homogeneous morphism of matrix factorizations preserving both gradings of M and satisfying $f \circ f \simeq f$. By definition of $\mathrm{hmf}_{R,w}$, M is homotopically finite. So, by Proposition 3.7 and Corollary 3.9, M_{es} is finitely generated over R . Note that f induces a morphism $f_{es} : M_{es} \rightarrow M_{es}$ such that $f_{es} \circ f_{es} \simeq f_{es}$. By the lifting idempotent property (Lemma 3.13), there is a morphism $g : M_{es} \rightarrow M_{es}$ preserving both gradings of M_{es} such that $g \simeq f_{es}$ and $g \circ g = g$. Now g give a decomposition of graded R -modules $M_{es} = gM_{es} \oplus (\mathrm{id} - g)M_{es}$. In particular, gM_{es} and $(\mathrm{id} - g)M_{es}$ are both projective modules over R . Recall that $R = \mathbb{C}[X_1, \dots, X_m]$ is a polynomial ring. The well known Quillen-Suslin Theorem tells us that any projective R -module is a free R -module. So gM_{es} and $(\mathrm{id} - g)M_{es}$ are finitely generated graded free R -modules. Since g is a morphism of matrix factorizations, the differential map on M_{es} induces differential maps on gM_{es} and $(\mathrm{id} - g)M_{es}$, which make them objects of $\mathrm{hmf}_{R,w}$ and the above decomposition a decomposition of graded matrix factorizations. Altogether, we have $M \simeq M_{es} = gM_{es} \oplus (\mathrm{id} - g)M_{es}$ as graded matrix factorizations. \square

Lemma 3.15. [18, Propositions 6] *$\mathrm{hmf}_{R,w}$ is locally finite dimensional.*

Proof. (Following [18].) Let M and M' be objects of $\mathrm{hmf}_{R,w}$. Then there exists finitely generated graded matrix factorizations \mathcal{M} and \mathcal{M}' over R of potential w such that $M \simeq \mathcal{M}$ and $M' \simeq \mathcal{M}'$. So $\mathrm{Hom}_{\mathrm{HMF}}(M, M') \cong \mathrm{Hom}_{\mathrm{HMF}}(\mathcal{M}, \mathcal{M}')$. Recall that R is a polynomial ring and, therefore, Noetherian. So $\mathrm{Hom}_{\mathrm{HMF}}(\mathcal{M}, \mathcal{M}')$ is finitely generated over R since $\mathrm{Hom}_R(\mathcal{M}, \mathcal{M}')$ is finitely generated over R . Let v_1, \dots, v_k be a finite set of homogeneous generators of $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ over R and $a = \min_{i=1, \dots, k} \deg v_i$. Then $\mathrm{Hom}_{\mathrm{hmf}}(M, M')$ is a quotient space of a subspace of the finite dimensional space

$$\left(\bigoplus_{i=1}^k \mathbb{C} \cdot v_i \right) \otimes_{\mathbb{C}} \left(\bigoplus_{j=0}^{-a} R^{(j)} \right),$$

where $R^{(j)}$ is the \mathbb{C} -subspace of R of homogeneous elements of degree j . Therefore, $\mathrm{Hom}_{\mathrm{hmf}}(M, M')$ is finite dimensional over \mathbb{C} . \square

The following is [18, Proposition 25] and follows easily from Lemmas 3.11, 3.14 and 3.15.

Proposition 3.16. [18, Proposition 25] *Assume that w is a homogeneous element of \mathfrak{J} with $\deg w = 2N + 2$. Then $\mathrm{hmf}_{R,w}$, $\mathrm{Ch}^b(\mathrm{hmf}_{R,w})$ and $\mathrm{hCh}^b(\mathrm{hmf}_{R,w})$ are all Krull-Schmidt.*

3.4. Yonezawa's lemma. Yonezawa [46] introduced a lemma about isomorphisms in a graded Krull-Schmidt category that is very useful in the proof of the invariance of the $\mathfrak{sl}(N)$ -homology of colored links. Next we review this lemma and show that it applies to $\mathrm{hmf}_{R,w}$ and $\mathrm{hCh}^b(\mathrm{hmf}_{R,w})$. Our statement of Yonezawa's lemma is slightly different from the original version in [46].

First, we recall a simple property of Krull-Schmidt categories.

Lemma 3.17. *Let \mathcal{C} be a Krull-Schmidt category, and A, B, C objects of \mathcal{C} . If $A \oplus C \cong B \oplus C$, then $A \cong B$.*

Proof. Decompose both sides of $A \oplus C \cong B \oplus C$ into direct sums of indecomposable objects and compare the components of these direct sums. \square

Definition 3.18. Let \mathcal{C} be an additive category, and $F : \mathcal{C} \rightarrow \mathcal{C}$ an autofunctor with inverse functor F^{-1} . We say that F is strongly non-periodic if, for any object A of \mathcal{C} and $k \in \mathbb{Z}$, $F^k(A) \cong A$ implies that either $A \cong 0$ or $k = 0$.

Denote by $\mathbb{Z}_{\geq 0}[F, F^{-1}]$ the ring of formal Laurant polynomials of F whose coefficients are non-negative integers. Each $G = \sum_{i=k}^l b_i F^i \in \mathbb{Z}_{\geq 0}[F, F^{-1}]$ admits a natural interpretation as an endofunctor on \mathcal{C} , that is, for any object A of \mathcal{C} ,

$$G(A) = \bigoplus_{i=k}^l \underbrace{(F^i(A) \oplus \cdots \oplus F^i(A))}_{b_i \text{ fold}}.$$

The following is Yonezawa's lemma.

Lemma 3.19. [46] *Let \mathcal{C} be a Krull-Schmidt category, and $F : \mathcal{C} \rightarrow \mathcal{C}$ a strongly non-periodic autofunctor. Suppose that A, B are objects of \mathcal{C} , and there exists a $G \in \mathbb{Z}_{\geq 0}[F, F^{-1}]$ such that $G \neq 0$ and $G(A) \cong G(B)$. Then $A \cong B$.*

Proof. For any objects C and C' , we say that they are in the same orbit if $C \cong F^k(C')$ for some $k \in \mathbb{Z}$. If C and C' are in the same orbit, and $C \not\cong 0$, then we can define a relative degree so that $\deg(C, C') = k$ if $C \cong F^k(C')$. This relative degree is well defined since F is strongly non-periodic.

Clearly, F preserves direct sum decompositions, maps isomorphic objects to isomorphic objects and maps indecomposable objects to indecomposable objects.

For any object C of \mathcal{C} , if $C \cong C_1 \oplus \cdots \oplus C_l$, where C_1, \dots, C_l are indecomposable objects of \mathcal{C} , then we say that l is the length of C and denote this by $L(C) = l$. Since \mathcal{C} is Krull-Schmidt, $L(C)$ is well defined. Clearly, $L(C) = L(F(C)) = L(F^k(C))$. More generally, for any $X \in \mathbb{Z}_{\geq 0}[F, F^{-1}]$, let $X(1) = X|_{F=1} \in \mathbb{Z}_{\geq 0}$. Then, $L(X(C)) = X(1)L(C)$ for any object C .

If $X \neq 0$, define the degree $\deg X$ of X to be the maximal k so that the coefficient of F^k in X is non-zero.

We prove the lemma by inducting on the length of A . If $L(A) = 0$, then $A \cong 0$ and $L(B)G(1) = L(A)G(1) = 0$. Since $G(1) > 0$, this implies that $L(B) = 0$ and, therefore, $B \cong 0$. So $A \cong B$. Assume that the lemma is true if $L(A) = l - 1$. Now suppose $L(A) = l$. Decompose $G(A) \cong G(B)$ into indecomposable objects and find all the orbits of indecomposable objects that appear in this decomposition. This gives us

$$G(A) \cong G(B) \cong G_1(C_1) \oplus \cdots \oplus G_k(C_k),$$

where G_1, \dots, G_k are non-zero elements of $\mathbb{Z}_{\geq 0}[F, F^{-1}]$, and C_1, \dots, C_k are indecomposable objects in different orbits. Thus,

$$\begin{aligned} A &= f_1(C_1) \oplus \cdots \oplus f_k(C_k), \\ B &= g_1(C_1) \oplus \cdots \oplus g_k(C_k), \end{aligned}$$

where $f_1, \dots, f_k, g_1, \dots, g_k$ are non-zero elements of $\mathbb{Z}_{\geq 0}[F, F^{-1}]$. Compare $\deg f_1$ and $\deg g_1$. Using the strong non-periodicity of F and the uniqueness of the decomposition into indecomposable objects, it is easy to conclude that $\deg f_1 + \deg G = \deg G_1 = \deg g_1 + \deg G$. So $\deg f_1 = \deg g_1 \triangleq d$. Define $\hat{f}_1 := f_1 - F^d$, $\hat{g}_1 := g_1 - F^d \in \mathbb{Z}_{\geq 0}[F, F^{-1}]$. Let

$$\begin{aligned} \hat{A} &= \hat{f}_1(C_1) \oplus f_2(C_2) \oplus \cdots \oplus f_k(C_k), \\ \hat{B} &= \hat{g}_1(C_1) \oplus g_2(C_2) \oplus \cdots \oplus g_k(C_k). \end{aligned}$$

Then

$$G(\hat{A}) \oplus (F^d \cdot G)(C_1) \cong G(A) \cong G(B) \cong G(\hat{B}) \oplus (F^d \cdot G)(C_1).$$

By Lemma 3.17, we have that $G(\hat{A}) \cong G(\hat{B})$. But $L(\hat{A}) = l - 1$. So, by induction hypothesis, $\hat{A} \cong \hat{B}$. Thus, $A \cong \hat{A} \oplus F^d(C_1) \cong \hat{B} \oplus F^d(C_1) \cong B$. \square

Note that the quantum grading shift functor $\{q\}$ on $\text{hmf}_{R,w}$ induces a quantum grading shift functor on $\text{hCh}^b(\text{hmf}_{R,w})$, which we again denote by $\{q\}$. The following is an easy consequence of Lemma 3.19 and is very useful in our argument.

Proposition 3.20. *Assume that w is a homogeneous element of \mathfrak{I} with $\deg w = 2N+2$. The functor $\{q\}$ is strongly non-periodic on both $\text{hmf}_{R,w}$ and $\text{hCh}^b(\text{hmf}_{R,w})$. Therefore, for any non-zero element $f(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$,*

- *if M and M' are objects of $\text{hmf}_{R,w}$, and $M\{f(q)\} \simeq M'\{f(q)\}$, then $M \simeq M'$;*
- *if C and C' are objects of $\text{hCh}^b(\text{hmf}_{R,w})$, and $C\{f(q)\} \simeq C'\{f(q)\}$, then $C \simeq C'$.*

Proof. We only need to show that $\{q\}$ is strongly non-periodic on both $\text{hmf}_{R,w}$ and $\text{hCh}^b(\text{hmf}_{R,w})$. The second half of the proposition follows from this and Proposition 3.16 and Lemma 3.19.

Let M be any object of $\text{hmf}_{R,w}$. Assume that $M \simeq M\{q^k\}$ for some $k \neq 0$. Without loss of generality, assume $k > 0$. Since M is homotopically finite, there exists a finitely generated object \mathcal{M} of $\text{hmf}_{R,w}$ such that $M \simeq \mathcal{M}$. So $\mathcal{M} \simeq \mathcal{M}\{q^k\}$, and, therefore $\mathcal{M} \simeq \mathcal{M}\{q^{ak}\}$ for any $a \in \mathbb{Z}_{>0}$. Let $\{e_1, \dots, e_n\}$ be a homogeneous basis for \mathcal{M} . Set $u = \max_{1 \leq i \leq n} \deg e_i$ and $l = \min_{1 \leq i \leq n} \deg e_i$. Note that l is the lowest grading for any non-vanishing homogeneous elements of \mathcal{M} . Choose an $a \in \mathbb{Z}_{>0}$ such that $ak > u - l$. Then $\mathcal{M} \simeq \mathcal{M}\{q^{ak}\}$ implies that there are homogeneous morphisms $f : \mathcal{M} \rightarrow \mathcal{M}$ of degree $-ak$ and $g : \mathcal{M} \rightarrow \mathcal{M}$ of degree ak such that $f \circ g \simeq g \circ f \simeq \text{id}_{\mathcal{M}}$. Note that $\deg f(e_i) \leq -ak + u < l \ \forall \ i = 1, \dots, n$, which implies that $f(e_i) = 0 \ \forall \ i = 1, \dots, n$. So $f = 0$ and, therefore, $\text{id}_{\mathcal{M}} \simeq 0$. Thus, $M \simeq \mathcal{M} \simeq 0$. This shows that $\{q\}$ is strongly non-periodic on $\text{hmf}_{R,w}$.

Note that any object of $\text{hCh}^b(\text{hmf}_{R,w})$ is isomorphic to an object whose underlying R -module is finitely generated, and any morphism of $\text{hCh}^b(\text{hmf}_{R,w})$ can be realized as a finite collection of homogeneous morphisms of graded matrix factorizations. So the above argument works for $\text{hCh}^b(\text{hmf}_{R,w})$ too. Thus, $\{q\}$ is also strongly non-periodic on $\text{hCh}^b(\text{hmf}_{R,w})$. \square

4. SYMMETRIC POLYNOMIALS

In this section, we review properties of symmetric polynomials used in this paper. Most of these materials can be found in e.g. [10, 11, 23, 24, 25, 47].

4.1. Notations and basic examples. In this paper, an alphabet means a finite collection of homogeneous indeterminants of degree 2. For an alphabet $\mathbb{X} = \{x_1, \dots, x_m\}$, we denote by $\mathbb{C}[\mathbb{X}]$ the polynomial ring $\mathbb{C}[x_1, \dots, x_m]$ and by $\text{Sym}(\mathbb{X})$ the ring of symmetric polynomials over \mathbb{C} in $\mathbb{X} = \{x_1, \dots, x_m\}$. Note that the grading on $\mathbb{C}[\mathbb{X}]$ (and $\text{Sym}(\mathbb{X})$) is given by $\deg x_j = 2$. For $k = 1, 2, \dots, m$, we denote by X_k the k -th elementary symmetric polynomial in \mathbb{X} , i.e.

$$X_k := \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

X_k is a homogeneous symmetric polynomial of degree $2k$. It is well known that X_1, \dots, X_m are independent and $\text{Sym}(\mathbb{X}) = \mathbb{C}[X_1, \dots, X_m]$. For convenience, we define

$$X_0 = 1 \text{ and } X_k = 0 \text{ if } k < 0 \text{ or } k > m.$$

There are two more relevant families of basic symmetric polynomials. The power sum symmetric polynomials $\{p_k(\mathbb{X}) \mid k \in \mathbb{Z}\}$ given by

$$p_k(\mathbb{X}) = \begin{cases} \sum_{i=1}^m x_i^k & \text{if } k \geq 0, \\ 0 & \text{if } k < 0, \end{cases}$$

and the complete symmetric polynomials $\{h_k(\mathbb{X}) \mid k \in \mathbb{Z}\}$ given by

$$h_k(\mathbb{X}) = \begin{cases} \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq m} x_{i_1} x_{i_2} \dots x_{i_k} & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Consider the generating functions of $\{X_k\}$, $\{p_k(\mathbb{X})\}$ and $\{h_k(\mathbb{X})\}$, i.e. the power series

$$\begin{aligned} E(t) &= \sum_{k=0}^m (-1)^k X_k t^k = \prod_{i=1}^m (1 - x_i t), \\ P(t) &= \sum_{k=0}^{\infty} p_{k+1}(\mathbb{X}) t^k = \sum_{i=1}^m \frac{x_i}{1 - x_i t}, \\ H(t) &= \sum_{k=0}^{\infty} h_k(\mathbb{X}) t^k = \prod_{i=1}^m (1 - x_i t)^{-1}. \end{aligned}$$

It is easy to see that $E(t) \cdot H(t) = 1$, $E'(t) \cdot H(t) = -P(t)$ and $E(t) \cdot P(t) = -E'(t)$. Hence,

$$(4.1) \quad \sum_{k=0}^l (-1)^k X_k h_{l-k}(\mathbb{X}) = \begin{cases} 0 & \text{if } l > 0, \\ 1 & \text{if } l = 0, \end{cases}$$

$$(4.2) \quad \sum_{k=1}^l (-1)^{k-1} k X_k h_{l-k}(\mathbb{X}) = p_l(\mathbb{X}),$$

$$(4.3) \quad \sum_{k=0}^{l-1} (-1)^k X_k p_{l-k}(\mathbb{X}) = (-1)^{l+1} l X_l,$$

where (4.3) is known as Newton's Identity.

Since $\text{Sym}(\mathbb{X}) = \mathbb{C}[X_1, \dots, X_m]$, $p_k(\mathbb{X})$ and $h_k(\mathbb{X})$ can be uniquely expressed as polynomials in X_1, \dots, X_m . In fact, we know that

$$(4.4) \quad p_k(\mathbb{X}) = p_{m,k}(X_1, \dots, X_m) = \begin{vmatrix} X_1 & X_2 & X_3 & \dots & X_{k-1} & kX_k \\ 1 & X_1 & X_2 & \dots & X_{k-2} & (k-1)X_{k-1} \\ 0 & 1 & X_1 & \dots & X_{k-3} & (k-2)X_{k-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & X_1 & 2X_2 \\ 0 & 0 & 0 & \dots & 1 & X_1 \end{vmatrix},$$

and

$$(4.5) \quad h_k(\mathbb{X}) = h_{m,k}(X_1, \dots, X_m) = \begin{vmatrix} X_1 & X_2 & X_3 & \cdots & X_{k-1} & X_k \\ 1 & X_1 & X_2 & \cdots & X_{k-2} & X_{k-1} \\ 0 & 1 & X_1 & \cdots & X_{k-3} & X_{k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & X_1 & X_2 \\ 0 & 0 & 0 & \cdots & 1 & X_1 \end{vmatrix}.$$

Equations (4.4) and (4.5) can be proved inductively using equations (4.1) and (4.3).

Lemma 4.1.

$$\frac{\partial}{\partial X_j} p_{m,l}(X_1, \dots, X_m) = (-1)^{j+1} l h_{m,l-j}(X_1, \dots, X_m).$$

Proof. Induct on l . If $l < j$, then both sides of the above equation are 0, and, therefore, the lemma is true. If $l = j$, by Newton's Identity (4.3), we have

$$p_{m,j} + \sum_{k=1}^{j-1} (-1)^k X_k p_{m,j-k} = (-1)^{j+1} j X_j.$$

Derive this equation by X_j , we get

$$\frac{\partial}{\partial X_j} p_{m,j} = (-1)^{j+1} j.$$

So the lemma is true when $l \leq j$.

Assume that $\exists n \geq j$ such that the lemma is true $\forall l \leq n$. Consider $l = n+1$. Use Newton's Identity (4.3) again. We get

$$p_{m,n+1} + \sum_{k=1}^n (-1)^k X_k p_{m,n+1-k} = (-1)^n (n+1) X_{n+1}.$$

Derive this equation by X_j , we get

$$\frac{\partial}{\partial X_j} p_{m,n+1} + (-1)^j p_{m,n+1-j} + \sum_{k=1}^n (-1)^k X_k \frac{\partial}{\partial X_j} p_{m,n+1-k} = 0.$$

So, by induction hypothesis,

$$\begin{aligned} \frac{\partial}{\partial X_j} p_{m,n+1} &= (-1)^{j+1} p_{m,n+1-j} + \sum_{k=1}^{n+1-j} (-1)^{k+j} (n+1-k) X_k h_{m,n+1-k-j} \\ (\text{by (4.2)}) &= \sum_{k=1}^{n+1-j} (-1)^{k+j} (n+1) X_k h_{m,n+1-k-j} \\ (\text{by (4.1)}) &= (-1)^{j+1} (n+1) h_{m,n+1-j}. \end{aligned}$$

□

4.2. Partitions and linear bases for the space of symmetric polynomials.

A partition λ is a finite non-increasing sequence of non-negative integers ($\lambda_1 \geq \cdots \geq \lambda_m$). Two partitions are considered the same if one can be changed into the other by adding or removing 0's at the end. For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$, write $|\lambda| = \sum_{j=1}^m \lambda_j$ and $l(\lambda) = \#\{j \mid \lambda_j > 0\}$. There is a natural ordering of partitions. For two partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_n)$, we say that $\lambda > \mu$ if the first non-vanishing $\lambda_j - \mu_j$ is positive.

The Ferrers diagram of a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ has λ_i boxes in the i -th row from the top with rows of boxes lined up on the left. Reflecting this Ferrers diagram across the northwest-southeast diagonal, we get the Ferrers diagram of another partition $\lambda' = (\lambda'_1 \geq \dots \geq \lambda'_k)$, which is called the conjugate of λ . Clearly, $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$ and $(\lambda')' = \lambda$.

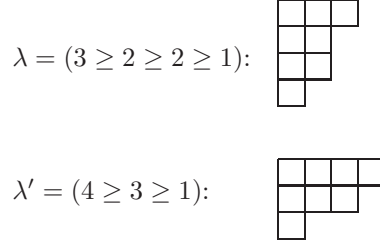


FIGURE 1. Ferrers diagrams of a partition and its conjugate

We are interested in partitions because they are used to index linear bases for the space of symmetric polynomials. We are particularly interested in two of such bases – the complete symmetric polynomials and the Schur polynomials.

Given an alphabet $\mathbb{X} = \{x_1, \dots, x_m\}$ of m indeterminants and a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ of length $l(\lambda) \leq m$, define

$$h_\lambda(\mathbb{X}) = h_{\lambda_1}(\mathbb{X}) \cdot h_{\lambda_2}(\mathbb{X}) \cdots h_{\lambda_m}(\mathbb{X}),$$

where $h_{\lambda_j}(\mathbb{X})$ is defined as in the previous subsection. $h_\lambda(\mathbb{X})$ is called the complete symmetric polynomial in \mathbb{X} associated to λ . This notion generalizes the definition of complete symmetric polynomials given in the previous subsection. It is known that the set $\{h_\lambda(\mathbb{X}) \mid l(\lambda) \leq m\}$ is a \mathbb{C} -linear basis for $\text{Sym}(\mathbb{X})$. In particular, $\{h_\lambda(\mathbb{X}) \mid l(\lambda) \leq m, |\lambda| = d\}$ is a \mathbb{C} -linear basis for the subspace of $\text{Sym}(\mathbb{X})$ of homogeneous symmetric polynomials of degree $2d$. (Recall that our degree is twice the usual degree.)

For the alphabet $\mathbb{X} = \{x_1, \dots, x_m\}$ and a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ of length $l(\lambda) \leq m$, the Schur polynomial in \mathbb{X} associated to λ is

$$S_\lambda(\mathbb{X}) = \frac{\begin{vmatrix} x_1^{\lambda_1+m-1} & x_1^{\lambda_2+m-2} & \cdots & x_1^{\lambda_{m-1}+1} & x_1^{\lambda_m} \\ x_2^{\lambda_1+m-1} & x_2^{\lambda_2+m-2} & \cdots & x_2^{\lambda_{m-1}+1} & x_2^{\lambda_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m-1}^{\lambda_1+m-1} & x_{m-1}^{\lambda_2+m-2} & \cdots & x_{m-1}^{\lambda_{m-1}+1} & x_{m-1}^{\lambda_m} \\ x_m^{\lambda_1+m-1} & x_m^{\lambda_2+m-2} & \cdots & x_m^{\lambda_{m-1}+1} & x_m^{\lambda_m} \end{vmatrix}}{\begin{vmatrix} x_1^{m-1} & x_1^{m-2} & \cdots & x_1 & 1 \\ x_2^{m-1} & x_2^{m-2} & \cdots & x_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m-1}^{m-1} & x_{m-1}^{m-2} & \cdots & x_{m-1} & 1 \\ x_m^{m-1} & x_m^{m-2} & \cdots & x_m & 1 \end{vmatrix}}.$$

Note that the denominator here is the Vandermonde polynomial, which equals $\prod_{i < j} (x_i - x_j)$. $S_\lambda(\mathbb{X})$ is also computed using the following formulas:

$$(4.6) \quad S_\lambda(\mathbb{X}) = \det(h_{\lambda_i - i + j}(\mathbb{X})) = \begin{vmatrix} h_{\lambda_1}(\mathbb{X}) & h_{\lambda_1+1}(\mathbb{X}) & \dots & h_{\lambda_1+m-1}(\mathbb{X}) \\ h_{\lambda_2-1}(\mathbb{X}) & h_{\lambda_2}(\mathbb{X}) & \dots & h_{\lambda_2+m-2}(\mathbb{X}) \\ \dots & \dots & \dots & \dots \\ h_{\lambda_m-m+1}(\mathbb{X}) & h_{\lambda_m-m+2}(\mathbb{X}) & \dots & h_{\lambda_m}(\mathbb{X}) \end{vmatrix},$$

and

$$(4.7) \quad S_\lambda(\mathbb{X}) = \det(X_{\lambda'_i - i + j}) = \begin{vmatrix} X_{\lambda'_1} & X_{\lambda'_1+1} & \dots & X_{\lambda'_1+k-1} \\ X_{\lambda'_2-1} & X_{\lambda'_2} & \dots & X_{\lambda'_2+k-2} \\ \dots & \dots & \dots & \dots \\ X_{\lambda'_k-k+1} & X_{\lambda'_k-k+2} & \dots & X_{\lambda'_k} \end{vmatrix},$$

where $\lambda' = (\lambda'_1 \geq \dots \geq \lambda'_k)$ is the conjugate of λ . In particular, for $j \geq 0$,

$$\begin{aligned} h_j(\mathbb{X}) &= S_{(j)}(\mathbb{X}), \\ X_j &= S_{\underbrace{(1 \geq 1 \geq \dots \geq 1)}_{j \text{ parts}}}(\mathbb{X}). \end{aligned}$$

The set $\{S_\lambda(\mathbb{X}) \mid l(\lambda) \leq m, |\lambda| = d\}$ is also a basis for the \mathbb{C} -space of homogeneous symmetric polynomials in \mathbb{X} of degree $2d$. (Again, recall that our degree is twice the usual degree.)

The above two bases for the space of symmetric polynomials are related by

$$(4.8) \quad h_\lambda(\mathbb{X}) = \sum_{\mu} K_{\mu\lambda} S_\mu(\mathbb{X}),$$

where $K_{\mu\lambda}$ is the Kostka number defined by

- $K_{\mu\lambda} = 0$ if $|\mu| \neq |\lambda|$;
- For partitions $\mu = (\mu_1 \geq \dots \geq \mu_m)$ and $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ with $|\mu| = |\lambda|$, $K_{\mu\lambda}$ is number of ways to fill boxes of the Ferrers diagram of μ with λ_1 1's, λ_2 2's, ..., λ_m m 's, such that the numbers in each row are nondecreasing from left to right, and the numbers in each column are strictly increasing from top to bottom.

Lemma 4.2. $K_{\lambda\lambda} = 1$ and $K_{\mu\lambda} = 0$ if $\lambda > \mu$, i.e. the first non-vanishing $\lambda_j - \mu_j$ is positive.

For an alphabet $\mathbb{X} = \{x_1, \dots, x_m\}$, there is also a notion of Schur polynomial in $-\mathbb{X}$, which will be useful in the next subsection. First, for any $j \in \mathbb{Z}$, define

$$h_j(-\mathbb{X}) = (-1)^j X_j.$$

More generally, for any partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ with $\lambda_1 \leq m$,

$$(4.9) \quad S_\lambda(-\mathbb{X}) = \det(h_{\lambda_i - i + j}(-\mathbb{X})) = \begin{vmatrix} h_{\lambda_1}(-\mathbb{X}) & h_{\lambda_1+1}(-\mathbb{X}) & \dots & h_{\lambda_1+n-1}(-\mathbb{X}) \\ h_{\lambda_2-1}(-\mathbb{X}) & h_{\lambda_2}(-\mathbb{X}) & \dots & h_{\lambda_2+n-2}(-\mathbb{X}) \\ \dots & \dots & \dots & \dots \\ h_{\lambda_n-n+1}(-\mathbb{X}) & h_{\lambda_n-n+2}(-\mathbb{X}) & \dots & h_{\lambda_n}(-\mathbb{X}) \end{vmatrix}.$$

If we write the Schur polynomials in \mathbb{X} as $S_\lambda(\mathbb{X}) = S_\lambda(x_1, \dots, x_m)$, then, by comparing (4.9) to (4.7), one can see that the Schur polynomials in $-\mathbb{X}$ is given by

$$(4.10) \quad S_\lambda(-\mathbb{X}) = S_{\lambda'}(-x_1, \dots, -x_m),$$

where λ' is the conjugate of λ .

See e.g. [11, Appendix A] and [24] for more on partitions and symmetric polynomials.

4.3. Partially symmetric polynomials. Let $\mathbb{X} = \{x_1, \dots, x_m\}$ and $\mathbb{Y} = \{y_1, \dots, y_n\}$ be two disjoint alphabets. Then $\mathbb{X} \cup \mathbb{Y}$ is also an alphabet. Denote by $\text{Sym}(\mathbb{X}|\mathbb{Y})$ the ring of polynomials in $\mathbb{X} \cup \mathbb{Y}$ over \mathbb{C} that are symmetric in \mathbb{X} and symmetric in \mathbb{Y} . Then $\text{Sym}(\mathbb{X} \cup \mathbb{Y})$, the ring of symmetric polynomials over \mathbb{C} in $\mathbb{X} \cup \mathbb{Y}$, is a subring of $\text{Sym}(\mathbb{X}|\mathbb{Y})$. Therefore, $\text{Sym}(\mathbb{X}|\mathbb{Y})$ is a $\text{Sym}(\mathbb{X} \cup \mathbb{Y})$ -module. The following theorem explains the structure of this module. (See [24, pages 16-19] for a detailed discussion.)

Theorem 4.3. [24, Proposition Gr5] *Let $\mathbb{X} = \{x_1, \dots, x_m\}$ and $\mathbb{Y} = \{y_1, \dots, y_n\}$ be two disjoint alphabets. Then $\text{Sym}(\mathbb{X}|\mathbb{Y})$ is a graded-free $\text{Sym}(\mathbb{X} \cup \mathbb{Y})$ -module.*

Denote by $\Lambda_{m,n}$ the set of partitions $\Lambda_{m,n} = \{\lambda \mid l(\lambda) \leq m, \lambda_1 \leq n\}$. Then

$$\{S_\lambda(\mathbb{X}) \mid \lambda \in \Lambda_{m,n}\} \text{ and } \{S_\lambda(-\mathbb{Y}) \mid \lambda \in \Lambda_{m,n}\}$$

are two homogeneous bases for the $\text{Sym}(\mathbb{X} \cup \mathbb{Y})$ -module $\text{Sym}(\mathbb{X}|\mathbb{Y})$.

Moreover, there is a unique $\text{Sym}(\mathbb{X} \cup \mathbb{Y})$ -module homomorphism

$$\zeta : \text{Sym}(\mathbb{X}|\mathbb{Y}) \rightarrow \text{Sym}(\mathbb{X} \cup \mathbb{Y}),$$

called the Sylvester operator, such that, for $\lambda, \mu \in \Lambda_{m,n}$,

$$\zeta(S_\lambda(\mathbb{X}) \cdot S_\mu(-\mathbb{Y})) = \begin{cases} 1 & \text{if } \lambda_j + \mu_{m+1-j} = n \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing Theorem 4.3 to equation (2.2), we get the following corollary.

Corollary 4.4. *Let $\mathbb{X} = \{x_1, \dots, x_m\}$ and $\mathbb{Y} = \{y_1, \dots, y_n\}$ be two disjoint alphabets. Then, as graded $\text{Sym}(\mathbb{X} \cup \mathbb{Y})$ -modules,*

$$\text{Sym}(\mathbb{X}|\mathbb{Y}) \cong \text{Sym}(\mathbb{X} \cup \mathbb{Y}) \left\{ \begin{bmatrix} m+n \\ n \end{bmatrix} \cdot q^{mn} \right\}.$$

More generally, given a collection $\{\mathbb{X}_1, \dots, \mathbb{X}_l\}$ of pairwise disjoint alphabets, we denote by $\text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_l)$ the ring of polynomials in $\mathbb{X}_1 \cup \dots \cup \mathbb{X}_l$ over \mathbb{C} that are symmetric in each \mathbb{X}_i , which is naturally a graded-free $\text{Sym}(\mathbb{X}_1 \cup \dots \cup \mathbb{X}_l)$ -module. Moreover,

$$\text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_l) \cong \text{Sym}(\mathbb{X}_1) \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} \text{Sym}(\mathbb{X}_l).$$

4.4. Cohomology ring of complex Grassmannian. Denote by $G_{m,N}$ the complex (m, N) -Grassmannian, i.e. the manifold of all complex m -dimensional subspaces of \mathbb{C}^N . The cohomology ring of $G_{m,N}$ is isomorphic to a quotient ring of a ring of symmetric polynomials. See e.g. [10, Lecture 6] for more.

Theorem 4.5. *Let \mathbb{X} be an alphabet of m independent indeterminants. Then $H^*(G_{m,N}; \mathbb{C}) \cong \text{Sym}(\mathbb{X}) / (h_{N+1-m}(\mathbb{X}), h_{N+2-m}(\mathbb{X}), \dots, h_N(\mathbb{X}))$ as graded \mathbb{C} -algebras. As a graded \mathbb{C} -linear space, $H^*(G_{m,N}; \mathbb{C})$ has a homogeneous basis*

$$\{S_\lambda(\mathbb{X}) \mid \lambda = (\lambda_1 \geq \dots \geq \lambda_m), \ l(\lambda) \leq m, \ \lambda_1 \leq N - m\}.$$

Under the above basis, the Poincaré duality of $H^(G_{m,N}; \mathbb{C})$ is given by a \mathbb{C} -linear trace map*

$$\text{Tr} : \text{Sym}(\mathbb{X}) / (h_{N+1-m}(\mathbb{X}), h_{N+2-m}(\mathbb{X}), \dots, h_N(\mathbb{X})) \rightarrow \mathbb{C}$$

satisfying

$$\mathrm{Tr}(S_\lambda(\mathbb{X}) \cdot S_\mu(\mathbb{X})) = \begin{cases} 1 & \text{if } \lambda_j + \mu_{m+1-j} = N - m \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing Theorem 4.5 to equation (2.2), we get the following corollary.

Corollary 4.6. *As graded \mathbb{C} -linear spaces,*

$$H^*(G_{m,N}; \mathbb{C}) \cong \mathbb{C} \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \cdot q^{m(N-m)} \right\},$$

where \mathbb{C} on the right hand side has grading 0.

5. MATRIX FACTORIZATIONS ASSOCIATED TO MOY GRAPHS

5.1. MOY graphs.

Definition 5.1. An abstract MOY graph is an oriented graph with each edge colored by a non-negative integer such that, for every vertex v with valence at least 2, the sum of integers coloring the edges entering v is equal to the sum of integers coloring the edges leaving v . We call this common sum the width of v .

A vertex of valence 1 in an abstract MOY graph is called an end point. An abstract MOY graph Γ is said to be closed if it has no end points.

An embedded MOY graph, or simply an MOY graph, Γ is an embedding of an abstract MOY graph into \mathbb{R}^2 such that, through each vertex v of Γ , there is a straight line L_v so that all the edges entering v enter through one side of L_v and all edges leaving v leave through the other side of L_v .

Remark 5.2. Before moving on, we should emphasize the following two points:

- (i) In this paper, an MOY graph means an embedded MOY graph.
- (ii) Every abstract MOY graph can not be realized as an (embedded) MOY graph.

Definition 5.3. A marking of an MOY graph Γ consists the following:

- (1) A finite collection of marked points on Γ such that
 - every edge of Γ has at least one marked point;
 - all the end points (vertices of valence 1) are marked;
 - none of the interior vertices (vertices of valence at least 2) is marked.
- (2) An assignment of pairwise disjoint alphabets to the marked points such that the alphabet associated to a marked point on an edge of color m has m independent indeterminants. (Recall that an alphabet is a finite collection of homogeneous indeterminants of degree 2.)

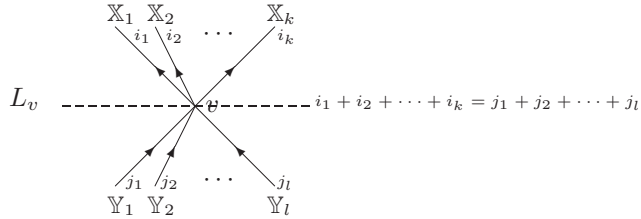


FIGURE 2.

5.2. The matrix factorization associated to an MOY graph. Recall that N is a fixed positive integer (i.e. the “ N ” in “ $\mathfrak{sl}(N)$ ”). For an MOY graph Γ with a marking, cut it open at the marked points. This gives a collection of marked MOY graphs, each of which is a star-shaped neighborhood of a vertex in G and is marked only at the endpoints. (If an edge of Γ has two or more marked points, then some of these pieces may be oriented arcs from one marked point to another. In this case, we consider such an arc as a neighborhood of an additional vertex of valence 2 in the middle of that arc.)

Let v be a vertex of Γ with coloring and marking around it given as in Figure 2. Set $m = i_1 + i_2 + \cdots + i_k = j_1 + j_2 + \cdots + j_l$ (the width of v .) Define

$$R = \text{Sym}(\mathbb{X}_1 | \cdots | \mathbb{X}_k | \mathbb{Y}_1 | \cdots | \mathbb{Y}_l).$$

Write $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_k$ and $\mathbb{Y} = \mathbb{Y}_1 \cup \cdots \cup \mathbb{Y}_l$. Denote by X_j the j -th elementary symmetric polynomial in \mathbb{X} and by Y_j the j -th elementary symmetric polynomial in \mathbb{Y} . For $j = 1, \dots, m$, define

$$U_j = \frac{p_{m,N+1}(Y_1, \dots, Y_{j-1}, X_j, \dots, X_m) - p_{m,N+1}(Y_1, \dots, Y_j, X_{j+1}, \dots, X_m)}{X_j - Y_j},$$

where $p_{m,N+1}$ is the polynomial given by equation (4.4) in Subsection 4.1. The matrix factorization associated to the vertex v is

$$C(v) = \begin{pmatrix} U_1 & X_1 - Y_1 \\ U_2 & X_2 - Y_2 \\ \vdots & \vdots \\ U_m & X_m - Y_m \end{pmatrix}_R \{q^{-\sum_{1 \leq s < t \leq k} i_s i_t}\},$$

whose potential is $\sum_{j=1}^m (X_j - Y_j)U_j = p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y})$, where $p_{N+1}(\mathbb{X})$ and $p_{N+1}(\mathbb{Y})$ are the $(N+1)$ -th power sum symmetric polynomials in \mathbb{X} and \mathbb{Y} . (See Subsection 4.1 for the definition.)

Remark 5.4. Since

$$\text{Sym}(\mathbb{X}|\mathbb{Y}) = \mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_m] = \mathbb{C}[X_1 - Y_1, \dots, X_m - Y_m, Y_1, \dots, Y_m],$$

it is clear that $\{X_1 - Y_1, \dots, X_m - Y_m\}$ is $\text{Sym}(\mathbb{X}|\mathbb{Y})$ -regular. By Theorem 4.3, R is a free $\text{Sym}(\mathbb{X}|\mathbb{Y})$ -module. It is then easy to see that $\{X_1 - Y_1, \dots, X_m - Y_m\}$ is also R -regular. So, by Lemma 2.18, the isomorphism type of $C(v)$ does not depend on the particular choice of U_1, \dots, U_m as long as they are homogeneous with the right degrees and the potential of $C(v)$ remains $\sum_{j=1}^m (X_j - Y_j)U_j = p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y})$. From now on, we will only specify our choice for U_1, \dots, U_m when it is used in the computation. Otherwise, we will simply denote them by $*$'s.

Definition 5.5.

$$C(\Gamma) := \bigotimes_v C(v),$$

where v runs through all the interior vertices of Γ (including those additional 2-valent vertices.) Here, the tensor product is done over the common end points. More precisely, for two sub-MOY graphs Γ_1 and Γ_2 of Γ intersecting only at (some of) their open end points, let $\mathbb{W}_1, \dots, \mathbb{W}_n$ be the alphabets associated to these common end points. Then, in the above tensor product, $C(\Gamma_1) \otimes C(\Gamma_2)$ is the tensor product $C(\Gamma_1) \otimes_{\text{Sym}(\mathbb{W}_1 | \dots | \mathbb{W}_n)} C(\Gamma_2)$.

$C(\Gamma)$ has a \mathbb{Z}_2 -grading and a quantum grading.

If Γ is closed, i.e. has no end points, then $C(\Gamma)$ is an object of $\text{hmf}_{\mathbb{C},0}$.

Assume Γ has end points. Let $\mathbb{E}_1, \dots, \mathbb{E}_n$ be the alphabets assigned to all end points of Γ , among which $\mathbb{E}_1, \dots, \mathbb{E}_k$ are assigned to exits and $\mathbb{E}_{k+1}, \dots, \mathbb{E}_n$ are assigned to entrances. Then the potential of $C(\Gamma)$ is

$$w = \sum_{i=1}^k p_{N+1}(\mathbb{E}_i) - \sum_{j=k+1}^n p_{N+1}(\mathbb{E}_j).$$

Let $R = \text{Sym}(\mathbb{E}_1 | \dots | \mathbb{E}_n)$. Although the alphabets assigned to all marked points on Γ are used in its construction, $C(\Gamma)$ is viewed as an object of $\text{hmf}_{R,w}$. Note that, in this case, w is a non-degenerate element of R .

We allow the MOY graph to be empty. In this case, we define

$$C(\emptyset) = \mathbb{C} \rightarrow 0 \rightarrow \mathbb{C},$$

where the \mathbb{Z}_2 -grading and the quantum grading of \mathbb{C} are both 0.

Lemma 5.6. *If Γ is an MOY graph, then the homotopy type of $C(\Gamma)$ does not depend on the choice of the marking.*

Proof. We only need to show that adding or removing an extra marked point corresponds to a homotopy of matrix factorizations preserving both gradings. This follows easily from Proposition 2.19. \square

Definition 5.7. Let Γ be an MOY graph with a marking.

- (i) If Γ is closed, i.e. has no open end points, then $C(\Gamma)$ is a chain complex. Denote by $H(\Gamma)$ the homology of $C(\Gamma)$. Note that $H(\Gamma)$ inherits both gradings of $C(\Gamma)$.
- (ii) If Γ has end points, let $\mathbb{E}_1, \dots, \mathbb{E}_n$ be the alphabets assigned to all end points of Γ , and $R = \text{Sym}(\mathbb{E}_1 | \dots | \mathbb{E}_n)$. Denote by $E_{i,j}$ the j -th elementary symmetric polynomial in \mathbb{E}_i and by \mathfrak{J} the maximal homogeneous ideal of R generated by $\{E_{i,j}\}$. Then $H(\Gamma)$ is defined to be $H_R(C(\Gamma))$, that is the homology of the chain complex $C(\Gamma)/\mathfrak{J} \cdot C(\Gamma)$. Clearly, $H(\Gamma)$ inherits both gradings of $C(\Gamma)$.

Note that (i) is a special case of (ii).

Lemma 5.8. *If Γ is an MOY graph with a vertex of width greater than N , then $C(\Gamma) \simeq 0$.*

Proof. Suppose the vertex v of Γ has width $m > N$. Then, by Newton's Identity (4.3), it is easy to check that, in the above construction, $U_{N+1} = (-1)^N(N+1)$ is a non-zero scalar. Apply the proof of Proposition 2.19 to the entry U_{N+1} in $C(\Gamma)$. One can see that $C(\Gamma) \simeq 0$. \square

Since rectangular partitions come up frequently in this paper, we introduce the following notations.

Definition 5.9. Denote by $\lambda_{m,n}$ the partition

$$\lambda_{m,n} := \underbrace{(n \geq \dots \geq n)}_{m \text{ parts}},$$

and $\Lambda_{m,n}$ the set of partitions

$$\Lambda_{m,n} := \{\mu \mid \mu \leq \lambda_{m,n}\} = \{\mu = (\mu_1 \geq \dots \geq \mu_m) \mid l(\mu) \leq m, \mu_1 \leq n\}.$$

The following is a generalization of [12, Proposition 2.4].

Lemma 5.10. *Let Γ be an MOY graph, and $\mathbb{X} = \{x_1, \dots, x_m\}$ an alphabet associated to a marked point on an edge of Γ of color m . Suppose that μ is a partition with $\mu > \lambda_{m, N-m}$, i.e. $\mu_1 - (N - m) > 0$. Then multiplication by $S_\mu(\mathbb{X})$ is a null-homotopic endomorphism of $C(\Gamma)$.*

Proof. Cut Γ open at all the marked points into local pieces, and let Γ' be a local piece containing the point marked by \mathbb{X} (as an end point.) Let $\mathbb{W}_1, \dots, \mathbb{W}_l$ be the alphabets marking other end points of Γ' . Then $C(\Gamma')$ is of the form

$$C(\Gamma') = \begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{k,0} & a_{k,1} \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{W}_1|\dots|\mathbb{W}_l)},$$

and has potential

$$\pm p_{N+1}(\mathbb{X}) + \sum_{i=1}^l \pm p_{N+1}(\mathbb{W}_i) = \sum_{j=1}^k a_{j,0} a_{j,1}.$$

Let X_j be the j -th elementary symmetric polynomial in \mathbb{X} . Derive the above equation by X_j . By Lemma 4.1, we get

$$\pm(N+1)h_{N+1-j}(\mathbb{X}) = \sum_{j=1}^k \left(\frac{\partial a_{j,0}}{\partial X_j} \cdot a_{j,1} + a_{j,0} \cdot \frac{\partial a_{j,1}}{\partial X_j} \right).$$

So $h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N-m+1}(\mathbb{X})$ are in the ideal $(a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1})$ of $\text{Sym}(\mathbb{X}|\mathbb{W}_1|\dots|\mathbb{W}_l)$. So, by Lemma 2.10, multiplications by these polynomials are null-homotopic endomorphisms of $C(\Gamma')$ and, by Lemma 2.9, of $C(\Gamma)$. By equation (4.6) and recursive relation (4.1), if $\mu > \lambda_{m, N-m}$, then $S_\mu(\mathbb{X})$ is in the ideal $(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N-m+1}(\mathbb{X}))$. So the multiplication by $S_\mu(\mathbb{X})$ is null-homotopic. \square

Lemma 5.11. *Let Γ be an MOY graph, and $\mathbb{E}_1, \dots, \mathbb{E}_n$ the alphabets assigned to all end points of Γ , among which $\mathbb{E}_1, \dots, \mathbb{E}_k$ are assigned to exits and $\mathbb{E}_{k+1}, \dots, \mathbb{E}_n$ are assigned to entrances. (Here we allow $n = 0$, i.e. Γ to be closed.) Write $R = \text{Sym}(\mathbb{E}_1|\dots|\mathbb{E}_n)$ and $w = \sum_{i=1}^k p_{N+1}(\mathbb{E}_i) - \sum_{j=k+1}^n p_{N+1}(\mathbb{E}_j)$. Then $C(\Gamma)$ is an object of $\text{hmf}_{R,w}$.*

Proof. Let $\mathbb{W}_1, \dots, \mathbb{W}_m$ be the alphabets assigned to interior marked points of Γ . Then $C(\Gamma)$ is a finitely generated Koszul matrix factorization over

$$\tilde{R} = \text{Sym}(\mathbb{W}_1|\dots|\mathbb{W}_m|\mathbb{E}_1|\dots|\mathbb{E}_n).$$

This implies that the quantum grading of $C(\Gamma)$ is bounded below. So, to show that $C(\Gamma)$ is an object of $\text{hmf}_{R,w}$, it remains to prove that $C(\Gamma)$ is homotopically finite. By Corollary 3.9, we only need to demonstrate that $H(\Gamma)$ is finite dimensional.

Let \mathfrak{I} be the maximal homogeneous ideal of $R = \text{Sym}(\mathbb{E}_1|\dots|\mathbb{E}_n)$. Then $C(\Gamma)/\mathfrak{I}C(\Gamma)$ is a chain complex of finitely generated modules over $R' = \text{Sym}(\mathbb{W}_1|\dots|\mathbb{W}_m)$. Note that R' is a polynomial ring and, therefore, Noetherian. So the homology of $C(\Gamma)/\mathfrak{I}C(\Gamma)$, i.e. $H(\Gamma)$, is also finitely generated over R' . But Lemma 5.10 implies that the action of R' on $H(\Gamma)$ factors through a finite dimensional quotient ring of R' . So $H(\Gamma)$ is finite dimensional over \mathbb{C} . \square

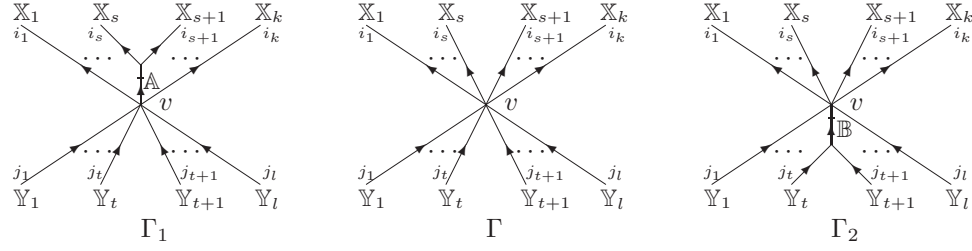


FIGURE 3.

Lemma 5.12. *Let Γ , Γ_1 and Γ_2 be MOY graphs shown in Figure 3. Then $C(\Gamma_1) \simeq C(\Gamma_2) \simeq C(\Gamma)$.*

Proof. We only prove that $C(\Gamma_1) \simeq C(\Gamma)$. The proof of $C(\Gamma_2) \simeq C(\Gamma)$ is similar. Set $m = i_1 + i_2 + \dots + i_k = j_1 + j_2 + \dots + j_l$. Let $R = \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_k | \mathbb{Y}_1 | \dots | \mathbb{Y}_l)$, and $\tilde{R} = \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_k | \mathbb{Y}_1 | \dots | \mathbb{Y}_l | \mathbb{A})$. Set $\mathbb{X} = \mathbb{X}_1 \cup \dots \cup \mathbb{X}_k$ and $\mathbb{Y} = \mathbb{Y}_1 \cup \dots \cup \mathbb{Y}_l$. Denote by X_j the j -th elementary symmetric polynomial in \mathbb{X} , by Y_j the j -th elementary symmetric polynomial in \mathbb{Y} , and by A_j the j -th elementary symmetric polynomial in \mathbb{A} . Moreover, denote by X'_j the j -th elementary symmetric polynomial in $\mathbb{X}_1 \cup \dots \cup \mathbb{X}_{s-1} \cup \mathbb{X}_{s+2} \cup \dots \cup \mathbb{X}_k$, and, for $i = s, s+1$, $X_{i,j}$ the j -th elementary symmetric polynomial in \mathbb{X}_i . Then

$$X_j = \sum_{p+q+r=j} X'_p X_{s,q} X_{s+1,r},$$

the j -th elementary symmetric polynomial in $\mathbb{X}_s \cup \mathbb{X}_{s+1}$ is

$$\sum_{p+q=j} X_{s,p} X_{s+1,q},$$

and the j -th elementary symmetric polynomial in $\mathbb{X}_1 \cup \dots \cup \mathbb{X}_{s-1} \cup \mathbb{X}_{s+2} \cup \dots \cup \mathbb{X}_k \cup \mathbb{A}$ is

$$\sum_{p+q=j} X'_p A_q.$$

Note that

$$\tilde{R} = R[A_1 - X_{s,1} - X_{s+1,1}, \dots, A_j - \sum_{p+q=j} X_{s,p} X_{s+1,q}, \dots, A_{i_s+i_{s+1}} - X_{s,i_s} X_{s+1,i_{s+1}}].$$

So, by Proposition 2.19,

$$\begin{aligned}
C(\Gamma_1) &\cong \begin{pmatrix} * & X'_1 + A_1 - Y_1 \\ \dots & \dots \\ * & \sum_{p+q=j} X'_p A_q - Y_j \\ \dots & \dots \\ * & X'_{m-i_s-i_{s+1}} A_{i_s+i_{s+1}} - Y_m \\ * & A_1 - X_{s,1} - X_{s+1,1} \\ \dots & \dots \\ * & A_j - \sum_{p+q=j} X_{s,p} X_{s+1,q} \\ \dots & \dots \\ * & A_{i_s+i_{s+1}} - X_{s,i_s} X_{s+1,i_{s+1}} \end{pmatrix}_{\tilde{R}} \{q^{-\sum_{1 \leq t_1 < t_2 \leq k} i_{t_1} i_{t_2}}\} \\
&\simeq \begin{pmatrix} * & X_1 - Y_1 \\ \dots & \dots \\ * & X_m - Y_m \end{pmatrix}_R \{q^{-\sum_{1 \leq t_1 < t_2 \leq k} i_{t_1} i_{t_2}}\} \\
&\cong C(\Gamma).
\end{aligned}$$

□

Lemma 5.12 implies that the matrix factorization associated to any MOY graph is homotopic to that associated to a trivalent MOY graph. So, theoretically, we do not lose any information by considering only the trivalent MOY graphs. But, in some cases, it is more convenient to use vertices of higher valence.

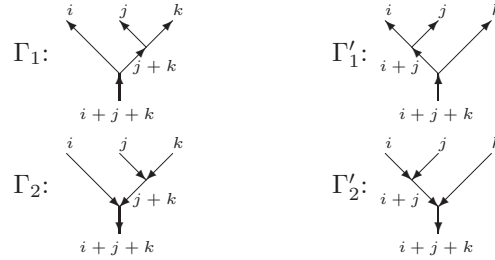


FIGURE 4.

Corollary 5.13. *Suppose that Γ_1 , Γ'_1 , Γ_2 and Γ'_2 are MOY graphs shown in Figure 4. Then $C(\Gamma_1) \simeq C(\Gamma'_1)$ and $C(\Gamma_2) \simeq C(\Gamma'_2)$.*

Proof. This is a special case of Lemma 5.12. □

5.3. Generalization of direct sum decomposition (II). We give a generalization of decomposition (II) first since it is useful in the generalization of decomposition (I). Its proof is a straightforward generalization of that in [18].

Theorem 5.14 (Direction Sum Decomposition II). *Suppose that Γ and Γ_1 are MOY graphs shown in Figure 5, where $n \geq m \geq 0$. Then*

$$C(\Gamma) \simeq C(\Gamma_1) \left\{ \begin{bmatrix} n \\ m \end{bmatrix} \right\}.$$

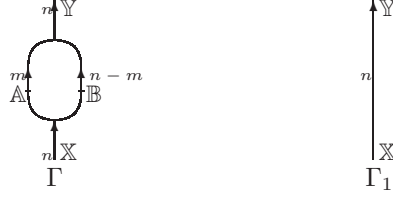


FIGURE 5.

Proof. Denote by X_j be j -th elementary symmetric polynomial in \mathbb{X} , and use similar notations for the other alphabets. Let $\mathbb{W} = \mathbb{A} \cup \mathbb{B}$. Then the j -th elementary symmetric polynomial in \mathbb{W} is

$$W_j = \sum_{p+q=j} A_p B_q.$$

By Theorem 4.3 and Corollary 4.4,

$$\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B}) = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W}) \{q^{m(n-m)} \begin{bmatrix} n \\ m \end{bmatrix}\}.$$

So

$$\begin{aligned} C(\Gamma) &\cong \left(\begin{array}{cc} * & Y_1 - W_1 \\ \dots & \dots \\ * & Y_n - W_n \\ * & W_1 - X_1 \\ \dots & \dots \\ * & W_n - X_m \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})} \{q^{-m(n-m)}\}, \\ &\cong \left(\begin{array}{cc} * & Y_1 - W_1 \\ \dots & \dots \\ * & Y_n - W_n \\ * & W_1 - X_1 \\ \dots & \dots \\ * & W_n - X_m \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W})} \left\{ \begin{bmatrix} n \\ m \end{bmatrix} \right\}, \\ &\cong \left(\begin{array}{cc} * & Y_1 - X_1 \\ \dots & \dots \\ * & Y_n - X_n \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y})} \left\{ \begin{bmatrix} n \\ m \end{bmatrix} \right\}, \\ &\cong C(\Gamma_1) \left\{ \begin{bmatrix} n \\ m \end{bmatrix} \right\}. \end{aligned}$$

where the homotopy is given by Proposition 2.19. \square

5.4. Generalization of direct sum decomposition (I). We prove a special case of the generalization first.

Lemma 5.15. *Suppose that Γ and Γ_1 are colored MOY graphs shown in Figure 6. Then $C(\Gamma) \simeq C(\Gamma_1) \langle N - m \rangle$.*

Proof. By Lemma 5.12, we have $C(\Gamma) \simeq C(\Gamma')$. So we only need to show that $C(\Gamma') \simeq C(\Gamma_1) \langle N - m \rangle$. We put markings on Γ' and Γ_1 as in Figure 6. Denote by X_j the j -th elementary symmetric polynomial in \mathbb{X} , and use similar notations for

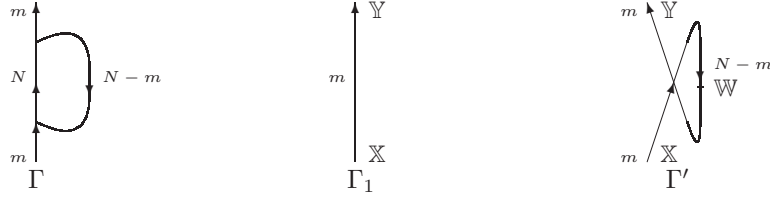


FIGURE 6.

the other alphabets. Write $\mathbb{A} = \mathbb{Y} \cup \mathbb{W}$ and $\mathbb{B} = \mathbb{X} \cup \mathbb{W}$. Then the j -th elementary symmetric polynomials in \mathbb{A} and \mathbb{B} are

$$\begin{aligned} A_j &= \sum_{p+q=j} Y_p W_q, \\ B_j &= \sum_{p+q=j} X_p W_q. \end{aligned}$$

Define

$$U_j = \frac{p_{N,N+1}(B_1, \dots, B_{j-1}, A_j, \dots, A_m) - p_{N,N+1}(B_1, \dots, B_j, A_{j+1}, \dots, A_m)}{A_j - B_j}.$$

Then

$$C(\Gamma') = \begin{pmatrix} U_1 & A_1 - B_1 \\ U_2 & A_2 - B_2 \\ \dots & \dots \\ U_N & A_N - B_N \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W})} \{q^{-m(N-m)}\}.$$

Using the relation $A_j - B_j = \sum_{p+q=j} (Y_p - X_p) W_q$ and, specially, $A_1 - B_1 = Y_1 - X_1$, we can inductive change the entries in the right column into $Y_1 - X_1, Y_2 - X_2, \dots, Y_m - X_m, 0, \dots, 0$ by the row operation given in Corollary 2.16. Note that these row operations do not change U_{m+1}, \dots, U_N in the left column. Thus,

$$C(\Gamma') \cong \begin{pmatrix} * & Y_1 - X_1 \\ \dots & \dots \\ * & Y_m - X_m \\ U_{m+1} & 0 \\ \dots & \dots \\ U_N & 0 \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W})} \{q^{-m(N-m)}\}.$$

Using Newton's Identity (4.3), one can verify that

$$p_{N,N+1}(A_1, \dots, A_N) = f_j + A_{N+1-j}(c_j A_j + g_j),$$

where f_j is a polynomial in $A_1, \dots, A_{N-j}, A_{N+2-j}, \dots, A_N$, and g_j is a polynomial in A_1, \dots, A_{j-1} , and

$$c_j = \begin{cases} (-1)^{N+1} \frac{N+1}{2}, & \text{if } N+1-j = j, \\ (-1)^{N+1} (N+1), & \text{if } N+1-j \neq j. \end{cases}$$

Therefore,

$$U_{N+1-j} = \begin{cases} (-1)^{N+1}(N+1)B_j + \alpha_j(B_1, \dots, B_{j-1}), & \text{if } N+1-j > j, \\ (-1)^{N+1}\frac{N+1}{2}(A_j + B_j) + \beta_j(B_1, \dots, B_{j-1}), & \text{if } N+1-j = j, \\ (-1)^{N+1}(N+1)A_j + \gamma_j(B_1, \dots, B_{N+1-j}, A_{N+1-j}, \dots, A_{j-1}), & \text{if } N+1-j < j, \end{cases}$$

where $\alpha_j, \beta_j, \gamma_j$ are polynomials in the given indeterminants.

So, for $j = 1, \dots, N-m$, U_{N+1-j} can be expressed as a polynomial

$$U_{N+1-j} = (-1)^{N+1}(N+1)W_j + u_j(X_1, \dots, X_m, Y_1, \dots, Y_m, W_1, \dots, W_{j-1}).$$

This implies that U_N, \dots, U_{m+1} are independent indeterminants over $\text{Sym}(\mathbb{X}|\mathbb{Y})$, and $\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W}) = \text{Sym}(\mathbb{X}|\mathbb{Y})[U_N, \dots, U_{m+1}]$. Hence, by Corollary 2.24,

$$\begin{aligned} & \left(\begin{array}{cc} * & Y_1 - X_1 \\ \dots & \dots \\ * & Y_m - X_m \\ U_{m+1} & 0 \\ \dots & \dots \\ U_N & 0 \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W})} \{q^{-m(N-m)}\} \\ & \simeq \left(\begin{array}{cc} * & Y_1 - X_1 \\ \dots & \dots \\ * & Y_m - X_m \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y})} \{q^{-m(N-m) + \sum_{j=m+1}^N (N+1 - \deg U_j)}\} \langle N-m \rangle \\ & \cong C(\Gamma_1) \langle N-m \rangle. \end{aligned}$$

Thus, $C(\Gamma) \cong C(\Gamma') \simeq C(\Gamma_1) \langle N-m \rangle$. □

The general case follows easily from Lemma 5.15.

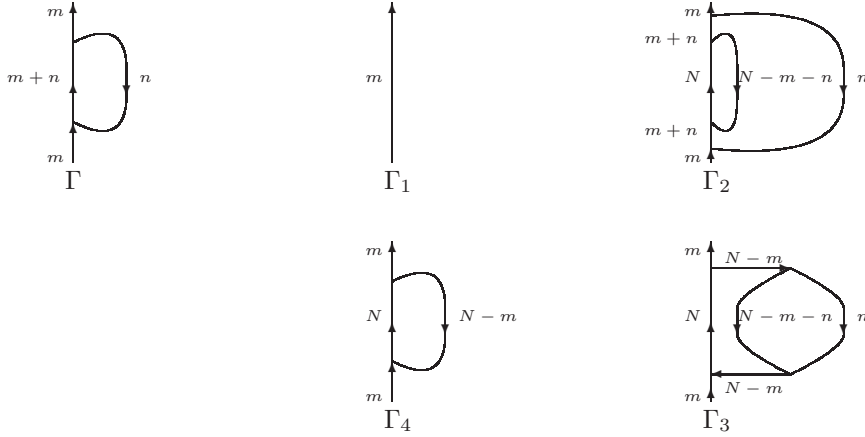


FIGURE 7.

Theorem 5.16 (Direction Sum Decomposition I). *Suppose that Γ and Γ_1 are colored MOY graphs shown in Figure 7. Then*

$$C(\Gamma) \simeq C(\Gamma_1) \left\{ \begin{bmatrix} N-m \\ n \end{bmatrix} \right\} \langle n \rangle.$$

Proof. Consider the colored MOY graphs in Figure 7. By Lemma 5.15, $C(\Gamma) \simeq C(\Gamma_2) \langle N - m - n \rangle$. By Corollary 5.13, $C(\Gamma_2) \simeq C(\Gamma_3)$. By Theorem 5.14, $C(\Gamma_3) \simeq C(\Gamma_4) \left[\begin{smallmatrix} N-m \\ n \end{smallmatrix} \right]$. And by Lemma 5.15 again, $C(\Gamma_4) \simeq C(\Gamma_1) \langle N - m \rangle$. Putting everything together, we get $C(\Gamma) \simeq C(\Gamma_1) \left\{ \left[\begin{smallmatrix} N-m \\ n \end{smallmatrix} \right] \right\} \langle n \rangle$. \square

6. CIRCLES

In this section, we study matrix factorizations associated to circles. The results will be useful in the next section.

6.1. Homotopy type. The following describes the homotopy type of the matrix factorization associated to a colored circle and follows easily from Direction Sum Decompositions I and II (Theorems 5.16 and 5.14.)

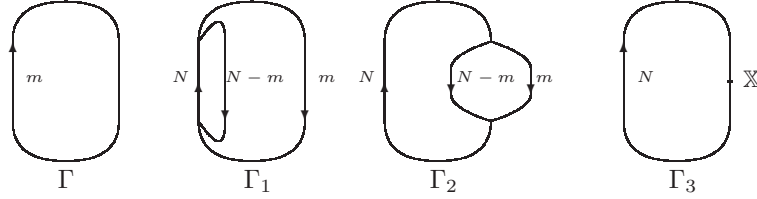


FIGURE 8.

Corollary 6.1. *If Γ is a circle colored by m , then $C(\Gamma) \simeq C(\emptyset) \left\{ \left[\begin{smallmatrix} N \\ m \end{smallmatrix} \right] \right\} \langle m \rangle$, where $C(\emptyset)$ is the matrix factorization $\mathbb{C} \rightarrow 0 \rightarrow \mathbb{C}$. As a consequence, $H(\Gamma) \cong C(\emptyset) \left\{ \left[\begin{smallmatrix} N \\ m \end{smallmatrix} \right] \right\} \langle m \rangle$.*

Proof. Consider Γ_3 in Figure 8 first, which is the special case when $m = N$. Note that, by Lemma 4.1,

$$C(\Gamma_3) \cong \left(\begin{array}{cc} \frac{\partial p_{N+1}(\mathbb{X})}{\partial X_1} & 0 \\ \cdots & \cdots \\ \frac{\partial p_{N+1}(\mathbb{X})}{\partial X_k} & 0 \\ \cdots & \cdots \\ \frac{\partial p_{N+1}(\mathbb{X})}{\partial X_N} & 0 \end{array} \right)_{\text{Sym}(\mathbb{X})} = \left(\begin{array}{cc} (N+1)h_N(\mathbb{X}) & 0 \\ \cdots & \cdots \\ (-1)^{k+1}(N+1)h_{N+1-k}(\mathbb{X}) & 0 \\ \cdots & \cdots \\ (-1)^{N+1}(N+1)h_1(\mathbb{X}) & 0 \end{array} \right)_{\text{Sym}(\mathbb{X})}$$

where X_k is the k -th elementary symmetric polynomial in \mathbb{X} . But

$$\text{Sym}(\mathbb{X}) = \mathbb{C}[h_1(\mathbb{X}), \dots, h_N(\mathbb{X})].$$

So, by applying Corollary 2.24 repeatedly, we get $C(\Gamma_3) \simeq C(\emptyset) \langle N \rangle$.

For the general case, using Theorem 5.14 and Lemma 5.15, we have

$$C(\Gamma) \simeq C(\Gamma_1) \langle N - m \rangle = C(\Gamma_2) \langle N - m \rangle \simeq C(\Gamma_3) \left\{ \left[\begin{smallmatrix} N \\ m \end{smallmatrix} \right] \right\} \langle N - m \rangle.$$

So $C(\Gamma) \simeq C(\emptyset) \left\{ \left[\begin{smallmatrix} N \\ m \end{smallmatrix} \right] \right\} \langle m \rangle$. \square

6.2. Module structure of the homology. Next we prove that, as a graded module, the homology of a colored circle is isomorphic to the cohomology, with a grading shift, of the corresponding complex Grassmannian. We need the following fact about symmetric polynomials to carry out our proof.

Proposition 6.2. *Let $\mathbb{X} = \{x_1, \dots, x_m\}$ be an alphabet with m independent indeterminants. If $n \geq m$, then the sequence $\{h_n(\mathbb{X}), h_{n-1}(\mathbb{X}), \dots, h_{n+1-m}(\mathbb{X})\}$ is $\text{Sym}(\mathbb{X})$ -regular. (c.f. Definition 2.17.)*

Proof. For $n, j \geq 1$, define an ideal $\mathcal{I}_{n,j}$ of $\text{Sym}(\mathbb{X})$ by $\mathcal{I}_{n,1} = \{0\}$ and $\mathcal{I}_{n,j} = (h_n(\mathbb{X}), h_{n-1}(\mathbb{X}), \dots, h_{n+2-j}(\mathbb{X}))$ for $j \geq 2$. For $1 \leq j \leq m \leq n$, let $P_{m,n,j}$ and $Q_{m,n,j}$ be the following statements:

- $P_{m,n,j}$: “ $h_{n+1-j}(\mathbb{X})$ is not a zero divisor of $\text{Sym}(\mathbb{X})/\mathcal{I}_{n,j}$.”
- $Q_{m,n,j}$: “ $X_m = x_1 \cdots x_m$ is not a zero divisor of $\text{Sym}(\mathbb{X})/\mathcal{I}_{n,j}$.”

We prove these two statements by induction for all m, n, j satisfying $1 \leq j \leq m \leq n$. Note that, by Definition 2.17, $\{h_n(\mathbb{X}), h_{n-1}(\mathbb{X}), \dots, h_{n+1-m}(\mathbb{X})\}$ is $\text{Sym}(\mathbb{X})$ -regular if $P_{m,n,j}$ is true for $1 \leq j \leq m$.

If $m = 1$, then $1 \leq j \leq m$ forces $j = 1$. Since $\mathcal{I}_{n,1} = \{0\}$, $P_{1,n,1}$ and $Q_{1,n,1}$ are trivially true for all $n \geq 1$. Assume that, for some $m \geq 2$, $P_{m-1,n,j}$ and $Q_{m-1,n,j}$ are true for all n, j with $1 \leq j \leq m-1 \leq n$. Consider $P_{m,n,j}$ and $Q_{m,n,j}$ for n, j satisfying $1 \leq j \leq m \leq n$.

(i) First, we prove $Q_{m,n,j}$ for all n, j with $1 \leq j \leq m \leq n$ by induction on j . When $j = 1$, $\mathcal{I}_{n,j} = \mathcal{I}_{n,1} = \{0\}$. So $Q_{m,n,1}$ is trivially true. Assume that $Q_{m,n,j-1}$ is true for some $j \geq 2$. Assume $g, g_n, \dots, g_{n+2-j} \in \text{Sym}(\mathbb{X})$ satisfy that

$$(6.1) \quad gX_m = \sum_{k=n+2-j}^n g_k h_k(\mathbb{X}).$$

Note that g, g_n, \dots, g_{n+2-j} are polynomials in X_1, \dots, X_m . We shall write

$$g = g(X_1, \dots, X_m), \quad g_n = g(X_1, \dots, X_m), \dots, \quad g_{n+2-j} = g(X_1, \dots, X_m).$$

Denote by X'_j the j -th elementary symmetric polynomial in $\mathbb{X}' = \{x_1, \dots, x_{m-1}\}$. Then $X_j|_{x_m=0} = X'_j$ and $h_j(\mathbb{X})|_{x_m=0} = h_j(\mathbb{X}')$. Plug $x_m = 0$ into (6.1). We get

$$\sum_{k=n+2-j}^n g_k(X'_1, \dots, X'_{m-1}, 0) h_k(\mathbb{X}') = 0.$$

Specially,

$$g_{n+2-j}(X'_1, \dots, X'_{m-1}, 0) h_{n+2-j}(\mathbb{X}') \in (h_n(\mathbb{X}'), h_{n-1}(\mathbb{X}'), \dots, h_{n+3-j}(\mathbb{X}')) \subset \text{Sym}(\mathbb{X}').$$

But Statement $P_{m-1,n,j-1}$ is true. So

$$g_{n+2-j}(X'_1, \dots, X'_{m-1}, 0) \in (h_n(\mathbb{X}'), h_{n-1}(\mathbb{X}'), \dots, h_{n+3-j}(\mathbb{X}')),$$

i.e.

$$\begin{aligned} g_{n+2-j}(X'_1, \dots, X'_{m-1}, 0) &= \sum_{k=n+3-j}^n \alpha_k(X'_1, \dots, X'_{m-1}) h_k(\mathbb{X}') \\ &= \sum_{k=n+3-j}^n \alpha_k(X'_1, \dots, X'_{m-1}) h_{m,k}(X'_1, \dots, X'_{m-1}, 0). \end{aligned}$$

Note that X'_1, \dots, X'_{m-1} are independent indeterminants over \mathbb{C} . So the above equation remains true when we replace X'_1, \dots, X'_{m-1} by any other variables. In particular,

$$g_{n+2-j}(X_1, \dots, X_{m-1}, 0) = \sum_{k=n+3-j}^n \alpha_k(X_1, \dots, X_{m-1}) h_{m,k}(X_1, \dots, X_{m-1}, 0),$$

which implies that there exists $\alpha \in \text{Sym}(\mathbb{X})$ such that

$$\begin{aligned} g_{n+2-j}(X_1, \dots, X_{m-1}, X_m) \\ &= \alpha X_m + \sum_{k=n+3-j}^n \alpha_k(X_1, \dots, X_{m-1}) h_{m,k}(X_1, \dots, X_{m-1}, X_m) \\ &= \alpha X_m + \sum_{k=n+3-j}^n \alpha_k(X_1, \dots, X_{m-1}) h_k(\mathbb{X}). \end{aligned}$$

Plug this into (6.1). We get

$$(g - \alpha h_{n+2-j}(\mathbb{X}))X_m = \sum_{k=n+3-j}^n (g_k + \alpha_k(X_1, \dots, X_{m-1}) h_{n+2-j}(\mathbb{X})) h_k(\mathbb{X}).$$

But $Q_{m,n,j-1}$ is true. So $g - \alpha h_{n+2-j}(\mathbb{X}) \in \mathcal{I}_{n,j-1}$ and, therefore, $g \in \mathcal{I}_{n,j}$. This proves $Q_{m,n,j}$. Thus, $Q_{m,n,j}$ is true for all n, j satisfying $1 \leq j \leq m \leq n$.

(ii) Now we prove $P_{m,n,j}$ for all n, j with $1 \leq j \leq m \leq n$.

Case A. $1 \leq j \leq m-1$. Assume that $h_{n+1-j}(\mathbb{X})$ is a zero divisor in $\text{Sym}(\mathbb{X})/\mathcal{I}_{n,j}$. Define

$$\Lambda = \{g \in \text{Sym}(\mathbb{X}) \mid g \text{ is homogeneous, } g \notin \mathcal{I}_{n,j}, gh_{n+1-j}(\mathbb{X}) \in \mathcal{I}_{n,j}\}.$$

Then $\Lambda \neq \emptyset$. Write $2\nu = \min_{g \in \Lambda} \deg g$. (Recall that we use the degree convention $\deg x_j = 2$.) Let g be such that $g \in \Lambda$ and $\deg g = 2\nu$. Then there exist $g_n, g_{n-1}, \dots, g_{n+2-j} \in \text{Sym}(\mathbb{X})$ such that $\deg g_k = 2(\nu + n + 1 - j - k)$ and

$$(6.2) \quad gh_{n+1-j}(\mathbb{X}) = \sum_{k=n+2-j}^n g_k h_k(\mathbb{X}).$$

Note that g, g_n, \dots, g_{n+2-j} are polynomials in X_1, \dots, X_m . We shall write

$$g = g(X_1, \dots, X_m), g_n = g(X_1, \dots, X_m), \dots, g_{n+2-j} = g(X_1, \dots, X_m).$$

In particular,

$$g = g(X_1, \dots, X_m) = \sum_{l=0}^{\lfloor \frac{\nu}{m} \rfloor} f_l(X_1, \dots, X_{m-1}) X_m^l,$$

where $f_l(X_1, \dots, X_{m-1}) \in \text{Sym}(\mathbb{X})$ is homogeneous of degree $2(\nu - lm)$.

Plug $x_m = 0$ into (6.2), we get

$$f_0(X'_1, \dots, X'_{m-1}) h_{n+1-j}(\mathbb{X}') = \sum_{k=n+2-j}^n g_k(X'_1, \dots, X'_{m-1}, 0) h_k(\mathbb{X}'),$$

where $\mathbb{X}' = \{x_1, \dots, x_{m-1}\}$ and X'_j is the j -th elementary symmetric polynomial in \mathbb{X}' . But $P_{m-1,n,j}$ is true since $1 \leq j \leq m-1 < n$. So

$$f_0(X'_1, \dots, X'_{m-1}) \in (h_n(\mathbb{X}'), h_{n-1}(\mathbb{X}'), \dots, h_{n+2-j}(\mathbb{X}')) \subset \text{Sym}(\mathbb{X}').$$

Thus,

$$\begin{aligned} f_0(X'_1, \dots, X'_{m-1}) &= \sum_{k=n+2-j}^n \alpha_k(X'_1, \dots, X'_{m-1}) h_k(\mathbb{X}') \\ &= \sum_{k=n+2-j}^n \alpha_k(X'_1, \dots, X'_{m-1}) h_{m,k}(X'_1, \dots, X'_{m-1}, 0), \end{aligned}$$

where $\alpha_k(X'_1, \dots, X'_{m-1}) \in \text{Sym}(\mathbb{X}')$ is homogeneous of degree $2(\nu - k)$. But X'_1, \dots, X'_{m-1} are independent indeterminants over \mathbb{C} . So the above equation remains true when we replace X'_1, \dots, X'_{m-1} by any other variables. In particular,

$$\begin{aligned} f_0(X_1, \dots, X_{m-1}) &= \sum_{k=n+2-j}^n \alpha_k(X_1, \dots, X_{m-1}) h_{m,k}(X_1, \dots, X_{m-1}, 0) \\ &= \alpha X_m + \sum_{k=n+2-j}^n \alpha_k(X_1, \dots, X_{m-1}) h_k(\mathbb{X}), \end{aligned}$$

where $\alpha \in \text{Sym}(\mathbb{X})$ is homogeneous of degree $2(\nu - m)$. Plug this in to (6.2). We get

$$\begin{aligned} &X_m(\alpha + \sum_{l=1}^{\lfloor \frac{\nu}{m} \rfloor} f_l(X_1, \dots, X_{m-1}) X_m^{l-1}) h_{n+1-j}(\mathbb{X}) \\ &= \sum_{k=n+2-j}^n (g_k - \alpha_k(X_1, \dots, X_{m-1}) h_{n+1-j}(\mathbb{X})) h_k(\mathbb{X}) \\ &\in \mathcal{I}_{n,j}. \end{aligned}$$

By $Q_{m,n,j}$, we have $(\alpha + \sum_{l=1}^{\lfloor \frac{\nu}{m} \rfloor} f_l(X_1, \dots, X_{m-1}) X_m^{l-1}) h_{n+1-j}(\mathbb{X}) \in \mathcal{I}_{n,j}$. But $\alpha + \sum_{l=1}^{\lfloor \frac{\nu}{m} \rfloor} f_l(X_1, \dots, X_{m-1}) X_m^{l-1}$ is homogeneous of degree $2(\nu - m) < 2\nu$. By the definition of ν , this implies that $\alpha + \sum_{l=1}^{\lfloor \frac{\nu}{m} \rfloor} f_l(X_1, \dots, X_{m-1}) X_m^{l-1} \in \mathcal{I}_{n,j}$. Then

$$g = X_m(\alpha + \sum_{l=1}^{\lfloor \frac{\nu}{m} \rfloor} f_l(X_1, \dots, X_{m-1}) X_m^{l-1}) + \sum_{k=n+2-j}^n \alpha_k(X_1, \dots, X_{m-1}) h_k(\mathbb{X}) \in \mathcal{I}_{n,j}.$$

This is a contradiction. So $P_{m,n,j}$ is true for all n, j such that $1 \leq j \leq m-1$, $m \leq n$.

Case B. $j = m$. We induct on n . Note that $h_m(\mathbb{X}), h_{m-1}(\mathbb{X}), \dots, h_1(\mathbb{X})$ are independent over \mathbb{C} , and $\text{Sym}(\mathbb{X}) = \mathbb{C}[h_m(\mathbb{X}), h_{m-1}(\mathbb{X}), \dots, h_1(\mathbb{X})]$. When $n = m$, $h_{n+1-m}(\mathbb{X}) = h_1(\mathbb{X})$ and $\text{Sym}(\mathbb{X})/\mathcal{I}_{m,m} \cong \mathbb{C}[h_1(\mathbb{X})]$. So $P_{m,m,m}$ is true. Assume that $P_{m,n-1,m}$ is true for some $n > m$. Suppose that $g_n, \dots, g_{n+1-m} \in \text{Sym}(\mathbb{X})$ satisfy

$$(6.3) \quad \sum_{k=n+1-m}^n g_k h_k(\mathbb{X}) = 0.$$

By equation (4.1), we have

$$h_n(\mathbb{X}) = \sum_{k=n-m}^{n-1} (-1)^{n-k+1} X_{n-k} h_k(\mathbb{X}).$$

Plug this into (6.3), we get

$$(6.4) \quad (-1)^{m+1} X_m g_n h_{n-m}(\mathbb{X}) + \sum_{k=n+1-m}^{n-1} (g_k + (-1)^{n-k+1} X_{n-k} g_n) h_k(\mathbb{X}) = 0$$

So $X_m g_n h_{n-m}(\mathbb{X}) \in \mathcal{I}_{n-1,m}$. Since $P_{m,n-1,m}$ and $Q_{m,n-1,m}$ are both true, this implies that $g_n \in \mathcal{I}_{n-1,m}$. Hence, there exist $\alpha_{n-1}, \dots, \alpha_{n+1-m} \in \text{Sym}(\mathbb{X})$ such that

$$(6.5) \quad g_n = \sum_{k=n+1-m}^{n-1} \alpha_k h_k(\mathbb{X}).$$

Plug this into (6.4), we get

$$\sum_{k=n+1-m}^{n-1} (g_k + (-1)^{n-k+1} X_{n-k} g_n + (-1)^{m+1} \alpha_k X_m h_{n-m}(\mathbb{X})) h_k(\mathbb{X}) = 0.$$

By $P_{m,n-1,m-1}$, this implies

$$g_{n+1-m} + (-1)^m X_{m-1} g_n + (-1)^{m+1} \alpha_{n+1-m} X_m h_{n-m}(\mathbb{X}) \in \mathcal{I}_{n-1,m-1}.$$

Comparing this with (6.5), we get

$$g_{n+1-m} + \alpha_{n+1-m} ((-1)^m X_{m-1} h_{n+1-m}(\mathbb{X}) + (-1)^{m+1} X_m h_{n-m}(\mathbb{X})) \in \mathcal{I}_{n-1,m-1}.$$

Therefore,

$$g_{n+1-m} + \alpha_{n+1-m} h_n(\mathbb{X}) = g_{n+1-m} + \alpha_{n+1-m} \sum_{k=n-m}^{n-1} (-1)^{n-k+1} X_{n-k} h_k(\mathbb{X}) \in \mathcal{I}_{n-1,m-1}.$$

Thus, $g_{n+1-m} \in \mathcal{I}_{n,m}$. This proves $P_{m,n,m}$. So $P_{m,n,m}$ is true for all $n \geq m$.

Combine *Case A* and *Case B*, we know that $P_{m,n,j}$ is true for all n, j such that $1 \leq j \leq m \leq n$.

(i) and (ii) show that $P_{m,n,j}$ and $Q_{m,n,j}$ are true for all m, n, j satisfying $1 \leq j \leq m \leq n$. \square

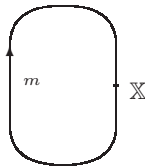


FIGURE 9.

Proposition 6.3. *If \bigcirc_m is the circle colored by m ($\leq N$) in Figure 9, then, as bigraded $\text{Sym}(\mathbb{X})$ -modules,*

$$H(\bigcirc_m) \cong \text{Sym}(\mathbb{X}) / (h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X})) \{q^{-m(N-m)}\} \langle m \rangle,$$

where \mathbb{X} is an alphabet of m indeterminants and

$$\text{Sym}(\mathbb{X})/(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X}))$$

has \mathbb{Z}_2 -grading 0.

In particular, as graded modules over $\text{Sym}(\mathbb{X})$, $H(\bigcirc_m) \cong H^*(G_{m,N})\{q^{-m(N-m)}\}$, where $G_{m,N}$ is the complex (m, N) -Grassmannian.

Proof. By definition,

$$C(\bigcirc_m) = \begin{pmatrix} U_1 & 0 \\ \cdots & \cdots \\ U_m & 0 \end{pmatrix}_{\text{Sym}(\mathbb{X})},$$

where $U_j = \frac{\partial}{\partial X_j} p_{m,N+1}(X_1, \dots, X_m)$. By Lemma 4.1, we know

$$U_j = (-1)^{j+1}(N+1)h_{m,N+1-j}(X_1, \dots, X_m).$$

Then, by Proposition 6.2, U_j is not a zero divisor in $\text{Sym}(\mathbb{X})/(U_1, \dots, U_{j-1})$. Thus, we can apply Corollary 2.25 successively to the rows of $C(\bigcirc_m)$ from top to bottom and conclude that

$$H(\bigcirc_m) \cong \text{Sym}(\mathbb{X})/(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X}))\{q^{-m(N-m)}\} \langle m \rangle.$$

The last statement in the proposition follows from Theorem 4.5. \square

From the above proposition, we know that $H(\bigcirc_m)$ is generated, as a $\text{Sym}(\mathbb{X})$ -module, by the homology class corresponding to

$$1 \in \text{Sym}(\mathbb{X})/(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X})).$$

We call this homology class the generating class and denote it by \mathfrak{G} .

6.3. Cycles representing the generating class. To understand the action of a morphism of matrix factorizations on the homology of a circle, we need to understand its action on the generating class \mathfrak{G} . In order to do that, we sometimes need to represent \mathfrak{G} by cycles in matrix factorizations associated to a circle. In particular, we will find such cycles in matrix factorizations associated to a circle with one or two marked points. To describe these cycles, we invoke the “ 1_ε ” notation introduced in Definition 2.4.

Lemma 6.4. *If \bigcirc_m is a circle colored by m ($\leq N$) with one marked point as shown in Figure 9, then, in*

$$C(\bigcirc_m) = \begin{pmatrix} U_1 & 0 \\ \cdots & \cdots \\ U_m & 0 \end{pmatrix}_{\text{Sym}(\mathbb{X})},$$

where $U_j = \frac{\partial}{\partial X_j} p_{m,N+1}(X_1, \dots, X_m)$, the element $1_{(1,1,\dots,1)}$ is a cycle representing (a non-zero scalar multiple of) the generating class $\mathfrak{G} \in H(\bigcirc_m)$.

Proof. Write

$$M_j = \begin{pmatrix} U_j & 0 \\ \cdots & \cdots \\ U_m & 0 \end{pmatrix}_{\text{Sym}(\mathbb{X})/(U_1, \dots, U_{j-1})}.$$

Then, the homology of Γ is computed by

$$\begin{aligned}
H(\bigcirc_m) &= H(M_1) \\
&\cong H(M_2)\{q^{N+1-\deg U_1}\}\langle 1 \rangle \\
&\cong \dots \\
&\cong H(M_m)\{q^{(m-1)(N+1)-\sum_{j=1}^{m-1} \deg U_j}\}\langle m-1 \rangle \\
&\cong \text{Sym}(\mathbb{X})/(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X}))\{q^{-m(N-m)}\}\langle m \rangle.
\end{aligned}$$

It is easy to see that $1_1 \in M_m$ represents \mathfrak{G} . Next, we use the method described in Remark 2.21 to inductively construct a cycle in $C(\bigcirc_m)$ representing the generating class. Assume, for some j , $1_{(1,1,\dots,1)} \in M_j$ is a cycle representing \mathfrak{G} . Note that $1_{(1,1,\dots,1)} \in M_{j-1}$ is mapped to $1_{(1,1,\dots,1)} \in M_j$ by the quasi-isomorphism $M_{j-1} \rightarrow M_j\{q^{N+1-\deg U_{j-1}}\}\langle 1 \rangle$. (Please see the proof of Proposition 2.20 for the definition of this quasi-isomorphism. Note that the setup there is slightly different – the construction there is modulo an entry in the right column there, but, here, U_{j-1} is in the left column.) But every entry in the right column of M_{j-1} is 0. So $d(1_{(1,1,\dots,1)}) = 0$, and therefore $1_{(1,1,\dots,1)}$ is a cycle representing \mathfrak{G} . This shows that $1_{(1,1,\dots,1)} \in M_1 = C(\bigcirc_m)$ is a cycle representing the generating class $\mathfrak{G} \in H(\bigcirc_m)$. \square

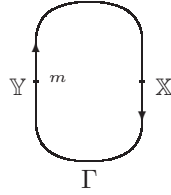


FIGURE 10.

Lemma 6.5. *Let \bigcirc_m be a circle colored by m ($\leq N$) with two marked points as shown in Figure 10. Use the definition*

$$C(\bigcirc_m) = \begin{pmatrix} U_1 & X_1 - Y_1 \\ \dots & \dots \\ U_m & X_m - Y_m \\ U_1 & Y_1 - X_1 \\ \dots & \dots \\ U_m & Y_m - X_m \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y})},$$

where X_j and Y_j are the j -th elementary symmetric polynomials in \mathbb{X} and in \mathbb{Y} , and $U_j \in \text{Sym}(\mathbb{X}|\mathbb{Y})$ is homogeneous of degree $2(N+1-j)$ and satisfies

$$\sum_{j=1}^m (X_j - Y_j) U_j = p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y}).$$

Then the element

$$\sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_m) \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + \sum_{j=1}^{m-1} (m-j)\varepsilon_j} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \in C(\bigcirc_m)$$

is a cycle representing (a non-zero scalar multiple of) the generating class $\mathfrak{G} \in H(\bigcirc_m)$.

Proof. Although this lemma can be proved by the method used in the previous lemma, the computation is far more complex. So here we use a different approach by considering morphisms of matrix factorizations. From Proposition 6.1, we have $H(\bigcirc_m) \cong C(\emptyset)\{ \begin{smallmatrix} N \\ m \end{smallmatrix} \} \langle m \rangle$. So the subspace of $H(\bigcirc_m)$ of elements of quantum degree $-m(N-m)$ is 1-dimensional over \mathbb{C} and is spanned by the generating class \mathfrak{G} . So, to prove the lemma, we only need to show that the above element of $C(\bigcirc_m)$ is a homogeneous cycle of quantum degree $-m(N-m)$ representing a non-zero homology class.

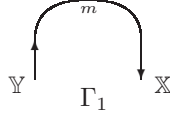


FIGURE 11.

Let Γ_1 be the oriented arc shown in Figure 11. Then, by lemmas 2.11, 2.12, and 2.13,

$$\mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1)) \cong C(\Gamma_1) \otimes_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})} C(\Gamma_1)_\bullet \cong C(\bigcirc_m)\{q^{m(N-m)}\} \langle m \rangle.$$

Consider the identity map $\mathrm{id} : C(\Gamma_1) \rightarrow C(\Gamma_1)$. It is clearly a morphism of matrix factorizations and, therefore, a cycle in $\mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1))$. If id is homotopic to 0, i.e. there exists $h \in \mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1))$ of \mathbb{Z}_2 -degree 1 such that $\mathrm{id} = d \circ h + h \circ d$. Then, for any cycle $f \in \mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1))$ of \mathbb{Z}_2 -degree i , we have

$$f = f \circ \mathrm{id} = f \circ (d \circ h + h \circ d) = (-1)^i (d \circ (f \circ h) - (-1)^{i+1} (f \circ h) \circ d),$$

which is a boundary element in $\mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1))$. This implies that the homology of $\mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1))$ is 0, which is a contradiction since $H(\bigcirc_m) \neq 0$. Thus id is a cycle representing a non-zero homology class. Under the above isomorphism, id is mapped to a homogeneous cycle in $C(\bigcirc_m)$ of quantum degree $-m(N-m)$ representing a non-zero homology class. Thus, the image of id is a cycle representing a non-zero scalar multiple of the generating class \mathfrak{G} .

Next, we check that the image of id is in fact the cycle given in this lemma. Under the isomorphism

$$\mathrm{Hom}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1)) \xrightarrow{\cong} C(\Gamma_1) \otimes_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})} C(\Gamma_1)_\bullet,$$

we have

$$\mathrm{id} \mapsto \sum_{\varepsilon \in I^m} 1_\varepsilon \otimes 1_\varepsilon^* \in C(\Gamma_1) \otimes_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})} C(\Gamma_1)_\bullet.$$

By Lemma 2.11, under the isomorphism (preserving both gradings)

$$C(\Gamma_1) \otimes_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})} C(\Gamma_1)_\bullet \xrightarrow{\cong} M_1 := \left(\begin{array}{cc} U_1 & X_1 - Y_1 \\ \vdots & \vdots \\ U_m & X_m - Y_m \\ Y_m - X_m & U_m \\ \vdots & \vdots \\ Y_1 - X_1 & U_1 \end{array} \right)_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y})},$$

we have

$$\sum_{\varepsilon \in I^m} 1_\varepsilon \otimes 1_\varepsilon^* \in C(\Gamma_1) \mapsto \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_m) \in I^m} 1_\varepsilon \otimes 1_{(\varepsilon_m, \dots, \varepsilon_1)} \in M_1.$$

By Lemma 2.12, under the isomorphism (preserving both gradings)

$$M_1 \xrightarrow{\cong} M_2 := \begin{pmatrix} U_1 & X_1 - Y_1 \\ \vdots & \vdots \\ U_m & X_m - Y_m \\ Y_1 - X_1 & U_1 \\ \vdots & \vdots \\ Y_m - X_m & U_m \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y})},$$

we have

$$\sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_m) \in I^m} 1_\varepsilon \otimes 1_{(\varepsilon_m, \dots, \varepsilon_1)} \mapsto \sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2}} 1_\varepsilon \otimes 1_\varepsilon \in M_2.$$

And, by lemmas 2.9 and 2.13, under the isomorphism (of \mathbb{Z}_2 -degree m and quantum degree $-m(N-m)$)

$$M_2 \rightarrow C(\bigcirc_m) = \begin{pmatrix} U_1 & X_1 - Y_1 \\ \vdots & \vdots \\ U_m & X_m - Y_m \\ U_1 & Y_1 - X_1 \\ \vdots & \vdots \\ U_m & Y_m - X_m \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y})},$$

we have

$$\sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2}} 1_\varepsilon \otimes 1_\varepsilon \mapsto \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_m) \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + \sum_{j=1}^{m-1} (m-j)\varepsilon_j} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \in C(\bigcirc_m).$$

Thus,

$$\sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_m) \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + \sum_{j=1}^{m-1} (m-j)\varepsilon_j} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \in C(\bigcirc_m)$$

is the image of $\text{id} \in \text{Hom}_{\text{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1))$ under the isomorphism

$$\text{Hom}_{\text{Sym}(\mathbb{X}|\mathbb{Y})}(C(\Gamma_1), C(\Gamma_1)) \rightarrow C(\bigcirc_m)$$

(of \mathbb{Z}_2 -degree m and quantum degree $-m(N-m)$.) \square

7. MORPHISMS INDUCED BY LOCAL CHANGES OF MOY GRAPHS

In this section, we establish several morphisms of matrix factorizations induced by certain local changes of MOY graphs, some of which has implicitly appeared in Sections 6 and 5. These morphisms are building blocks of more complex morphisms in Direct Sum Decompositions (III-V) and in the chain complexes of link diagrams.

7.1. Terminology. Most morphisms defined in the rest of this paper are defined only up to homotopy and scaling by a non-zero scalar. To simplify our exposition, we introduce the following notations.

Definition 7.1. Suppose that V is a linear space over \mathbb{C} and $u, v \in V$. We write $u \propto v$ if $\exists c \in \mathbb{C} \setminus \{0\}$ such that $u = c \cdot v$.

Suppose that W is a chain complex over a \mathbb{C} -algebra and u, v are cycles in W , we write $u \approx v$ if $\exists c \in \mathbb{C} \setminus \{0\}$ such that u is homologous to $c \cdot v$. In particular, if M, M' are matrix factorizations of the same potential over a graded commutative unital \mathbb{C} -algebra and $f, g : M \rightarrow M'$ are morphisms of matrix factorizations, we write $f \approx g$ if $\exists c \in \mathbb{C} \setminus \{0\}$ such that $f \simeq c \cdot g$.

Let Γ_1, Γ_2 be two colored MOY graphs with a one-to-one correspondence F between their end points such that

- every exit corresponds to an exit, and every entrance corresponds to an entrance,
- edges adjacent to corresponding end points have the same color.

Mark Γ_1, Γ_2 so that every pair of corresponding end points are assigned the same alphabet, and alphabets associated to internal marked points are pairwise disjoint. Let $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n$ be the alphabets assigned to the end points of Γ_1, Γ_2 .

Definition 7.2.

$$\text{Hom}_F(C(\Gamma_1), C(\Gamma_2)) := \text{Hom}_{\text{Sym}(\mathbb{X}_1|\mathbb{X}_2|\dots|\mathbb{X}_n)}(C(\Gamma_1), C(\Gamma_2)),$$

which is a \mathbb{Z}_2 -graded chain complex, where the \mathbb{Z}_2 -grading is induced by the \mathbb{Z}_2 -gradings of $C(\Gamma_1), C(\Gamma_2)$. The quantum gradings of $C(\Gamma_1), C(\Gamma_2)$ induce a quantum pregrading on $\text{Hom}_F(C(\Gamma_1), C(\Gamma_2))$.

Denote by $\text{Hom}_{\text{HMF}, F}(C(\Gamma_1), C(\Gamma_2))$ the homology of $\text{Hom}_F(C(\Gamma_1), C(\Gamma_2))$, i.e. the module of homotopy classes of morphisms from $C(\Gamma_1)$ to $C(\Gamma_2)$. It inherits the \mathbb{Z}_2 -grading from $\text{Hom}_F(C(\Gamma_1), C(\Gamma_2))$. And the quantum pregrading of $\text{Hom}_F(C(\Gamma_1), C(\Gamma_2))$ induces a quantum grading on $\text{Hom}_{\text{HMF}, F}(C(\Gamma_1), C(\Gamma_2))$. (See Lemmas 2.32 and 5.11.)

We drop F from the above notations if it is clear from the context.

Lemma 7.3. $\text{Hom}_{\text{HMF}, F}(C(\Gamma_1), C(\Gamma_2))$ does not depend on the choice of markings.

Proof. This lemma follows easily from Proposition 2.19 and Corollary 2.22. \square

7.2. Bouquet move. First we recall the homotopy equivalence induced by the bouquet moves in Figure 12. From Corollary 5.13, we know bouquet moves induce homotopy equivalence. In this subsection, we show that, up to homotopy and scaling, a bouquet move induces a unique homotopy equivalence.

Lemma 7.4. Suppose that $\Gamma_1, \Gamma'_1, \Gamma_2$ and Γ'_2 are MOY graphs shown in Figure 12. Then, as $\mathbb{Z}_2 \oplus \mathbb{Z}$ -graded vector spaces over \mathbb{C} ,

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma'_1)) \cong \text{Hom}_{\text{HMF}}(C(\Gamma_2), C(\Gamma'_2)) \\ & \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ i+j+k \end{bmatrix} \begin{bmatrix} i+j+k \\ k \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} q^{(i+j+k)(N-i-j-k)+ij+jk+ki} \right\}. \end{aligned}$$

In particular, the subspaces of the above spaces of homogeneous elements of quantum degree 0 is 1-dimensional.

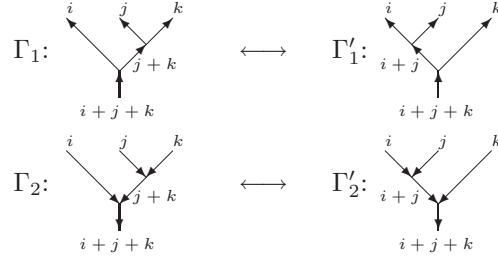


FIGURE 12.

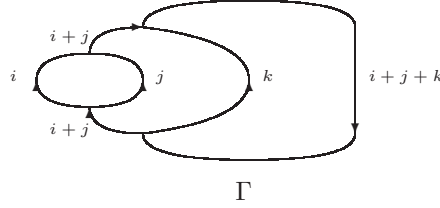


FIGURE 13.

Proof. We only compute $\text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma'_1))$. The computation of $\text{Hom}_{\text{HMF}}(C(\Gamma_2), C(\Gamma'_2))$ is similar. By Corollary 5.13 and Lemmas 2.11, 2.12, 2.13, one can see that

$$\begin{aligned} \text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma'_1)) &\cong \text{Hom}_{\text{HMF}}(C(\Gamma'_1), C(\Gamma'_1)) \\ &\cong H(\Gamma) \langle i+j+k \rangle \{ q^{(i+j+k)(N-i-j-k)+ij+jk+ki} \}, \end{aligned}$$

where Γ is the MOY graph in Figure 13. Using Decomposition (II) (Theorems 5.14) and Corollary 6.1, we have that

$$H(\Gamma) \cong C(\emptyset) \langle i+j+k \rangle \left\{ \begin{bmatrix} N \\ i+j+k \end{bmatrix} \begin{bmatrix} i+j+k \\ k \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} \right\}.$$

The lemma follows from these isomorphisms. \square

Remark 7.5. From Corollary 5.13 and Lemma 7.4, one can see that, up to homotopy and scaling, a bouquet move induces a unique homotopy equivalence. In the rest of this paper, we usually denote such a homotopy equivalence by h .

7.3. Circle creation and annihilation.

Lemma 7.6. *Let \bigcirc_m be a circle colored by m . Then, as $\mathbb{Z}_2 \oplus \mathbb{Z}$ -graded vector spaces over \mathbb{C} ,*

$$\text{Hom}_{\text{HMF}}(C(\bigcirc_m), C(\emptyset)) \cong \text{Hom}_{\text{HMF}}(C(\emptyset), C(\bigcirc_m)) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \right\} \langle m \rangle,$$

where $C(\emptyset)$ is the matrix factorization $\mathbb{C} \rightarrow 0 \rightarrow \mathbb{C}$.

Proof. The natural isomorphism $\text{Hom}(C(\emptyset), C(\bigcirc_m)) \cong C(\bigcirc_m)$ is an isomorphism of matrix factorizations preserving both gradings. So, by Corollary 6.1,

$$\text{Hom}_{\text{HMF}}(C(\emptyset), C(\bigcirc_m)) \cong H(\bigcirc_m) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \right\} \langle m \rangle.$$

By Corollary 6.1, we have

$$\mathrm{Hom}_{HMF}(C(\bigcirc_m), C(\emptyset)) \cong \mathrm{Hom}_{\mathbb{C}}(C(\emptyset)\left\{\begin{bmatrix} N \\ m \end{bmatrix}\right\}\langle m \rangle, C(\emptyset)) \cong C(\emptyset)\left\{\begin{bmatrix} N \\ m \end{bmatrix}\right\}\langle m \rangle.$$

□

By Lemma 7.6, the subspaces of $\mathrm{Hom}_{HMF}(C(\emptyset), C(\bigcirc_m))$ and $\mathrm{Hom}_{HMF}(C(\bigcirc_m), C(\emptyset))$ of elements of quantum degree $-m(N-m)$ are 1-dimensional. This leads to the following definitions, which generalize the corresponding definitions in [18].

Definition 7.7. Let \bigcirc_m be a circle colored by m . Associate to the circle creation a homogeneous morphism

$$\iota : C(\emptyset)(\cong \mathbb{C}) \rightarrow C(\bigcirc_m)$$

of quantum degree $-m(N-m)$ not homotopic to 0.

Associate to the circle annihilation a homogeneous morphism

$$\epsilon : C(\bigcirc_m) \rightarrow C(\emptyset)(\cong \mathbb{C})$$

of quantum degree $-m(N-m)$ not homotopic to 0.

By Lemma 7.6, ι and ϵ are unique up to homotopy and scaling. Both of them have \mathbb{Z}_2 -degree m . By the natural isomorphism $\mathrm{Hom}(C(\emptyset), C(\bigcirc_m)) \cong C(\bigcirc_m)$, it is easy to see that

$$(7.1) \quad \iota(1) \approx \mathfrak{G},$$

where \mathfrak{G} is the generating class of $H(\bigcirc_m)$.

Mark \bigcirc_m by a single alphabet \mathbb{X} . From the proof of Proposition 6.3, we know that there is a $\mathrm{Sym}(\mathbb{X})$ -linear projection

$$P : C(\bigcirc_m) \rightarrow \mathrm{Sym}(\mathbb{X})/(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X}))\{q^{-m(N-m)}\}\langle m \rangle$$

satisfying $P(\mathfrak{G}) = 1$. By Corollary 2.26 and Remark 2.23, P induces a quasi-isomorphism

$$\begin{aligned} & \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}(\mathbb{X})/(h_N(\mathbb{X}), h_{N-1}(\mathbb{X}), \dots, h_{N+1-m}(\mathbb{X}))\{q^{-m(N-m)}\}\langle m \rangle, \mathbb{C}) \\ & \xrightarrow{P^\#} \mathrm{Hom}(C(\bigcirc_m), C(\emptyset)). \end{aligned}$$

Recall that, by Theorem 4.5, there is a \mathbb{C} -linear trace map

$$\mathrm{Tr} : \mathrm{Sym}(\mathbb{X})/(h_{N+1-m}(\mathbb{X}), h_{N+2-m}(\mathbb{X}), \dots, h_N(\mathbb{X})) \rightarrow \mathbb{C}$$

satisfying

$$\mathrm{Tr}(S_\lambda(\mathbb{X}) \cdot S_\mu(\mathbb{X})) = \begin{cases} 1 & \text{if } \lambda_j + \mu_{m+1-j} = N-m \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda, \mu \in \Lambda_{m, N-m}$ and $S_\lambda(\mathbb{X})$ is the Schur polynomial in \mathbb{X} associated to the partition λ . Note that $P^\#(\mathrm{Tr}) = \mathrm{Tr} \circ P : C(\bigcirc_m) \rightarrow C(\emptyset)$ is homogeneous of \mathbb{Z}_2 -grading m and quantum grading $-m(N-m)$, and

$$(7.2) \quad P^\#(\mathrm{Tr})(S_\lambda(\mathbb{X}) \cdot S_\mu(\mathbb{X}) \cdot \mathfrak{G}) = \begin{cases} 1 & \text{if } \lambda_j + \mu_{m+1-j} = N-m \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

So $P^\#(\mathrm{Tr})$ is homotopically non-trivial. Therefore,

$$(7.3) \quad \epsilon \approx P^\#(\mathrm{Tr}) = \mathrm{Tr} \circ P.$$

Corollary 7.8. Denote by $\mathbf{m}(S_\lambda(\mathbb{X}))$ the morphism $C(\bigcirc_m) \rightarrow C(\bigcirc_m)$ induced by multiplication by $S_\lambda(\mathbb{X})$. Then, for any $\lambda, \mu \in \Lambda_{m, N-m}$,

$$\epsilon \circ \mathbf{m}(S_\lambda(\mathbb{X})) \circ \mathbf{m}(S_\mu(\mathbb{X})) \circ \iota \approx \begin{cases} \text{id}_{C(\emptyset)} & \text{if } \lambda_j + \mu_{m+1-j} = N - m \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This corollary follows easily from (7.1), (7.2) and (7.3). \square

7.4. Edge splitting and merging. Let Γ_0 and Γ_1 be the MOY graphs in Figure 14. We call the change $\Gamma_0 \rightsquigarrow \Gamma_1$ an edge splitting and the change $\Gamma_1 \rightsquigarrow \Gamma_0$ an edge merging. In this subsection, we define morphisms ϕ and $\bar{\phi}$ associated to edge splitting and merging.

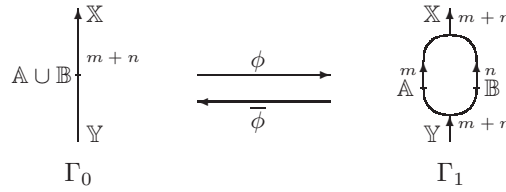


FIGURE 14.

Lemma 7.9. Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 14. Then, as bigraded vector spaces over \mathbb{C} ,

$$\text{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1)) \cong \text{Hom}_{HMF}(C(\Gamma_1), C(\Gamma_0)) \cong C(\emptyset) \{ q^{(N-m-n)(m+n)} \begin{bmatrix} N \\ m+n \end{bmatrix} \begin{bmatrix} m+n \\ m \end{bmatrix} \}.$$

In particular, the lowest quantum gradings of the above spaces are $-mn$, and the subspaces of these spaces of homogeneous elements of quantum grading $-mn$ are all 1-dimensional.

Proof. By Theorem 5.14, $C(\Gamma_1) \simeq C(\Gamma_0) \{ \begin{bmatrix} m+n \\ m \end{bmatrix} \}$. So it is easy to see that

$$\text{Hom}(C(\Gamma_0), C(\Gamma_1)) \simeq \text{Hom}(C(\Gamma_0), C(\Gamma_0)) \{ \begin{bmatrix} m+n \\ m \end{bmatrix} \} \simeq \text{Hom}(C(\Gamma_1), C(\Gamma_0)).$$

Denote by \bigcirc_{m+n} the circle colored by $m+n$. Then, from the proof of Lemma 6.5, we have

$$\begin{aligned} \text{Hom}(C(\Gamma_0), C(\Gamma_0)) &\cong C(\bigcirc_{m+n}) \{ q^{(N-m-n)(m+n)} \} \langle m+n \rangle \\ &\simeq C(\emptyset) \{ q^{(N-m-n)(m+n)} \begin{bmatrix} N \\ m+n \end{bmatrix} \}, \end{aligned}$$

and the lemma follows. \square

Definition 7.10. Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 14. Associate to the edge splitting a homogeneous morphism

$$\phi : C(\Gamma_0) \rightarrow C(\Gamma_1)$$

of quantum degree $-mn$ not homotopic to 0.

Associate to the edge merging a homogeneous morphism

$$\bar{\phi} : C(\Gamma_1) \rightarrow C(\Gamma_0)$$

of quantum degree $-mn$ not homotopic to 0.

The morphisms ϕ and $\bar{\phi}$ are well defined up to scaling and homotopy. By Lemma 7.9, both of them have \mathbb{Z}_2 -grading 0. It is not hard to find explicit forms of these morphisms. In fact, ϕ is the composition

$$C(\Gamma_0) \xrightarrow{\text{id}} C(\Gamma_0)\{q^{-mn}\} \hookrightarrow C(\Gamma_0)\left\{\begin{bmatrix} m+n \\ m \end{bmatrix}\right\} \xrightarrow{\simeq} C(\Gamma_1),$$

and $\bar{\phi}$ is the composition

$$C(\Gamma_1) \xrightarrow{\simeq} C(\Gamma_0)\left\{\begin{bmatrix} m+n \\ m \end{bmatrix}\right\} \twoheadrightarrow C(\Gamma_0)\{q^{mn}\} \xrightarrow{\text{id}} C(\Gamma_0),$$

where \hookrightarrow and \twoheadrightarrow are the natural inclusion and projection maps.

More precisely, from the proof of Theorem 5.14, we know that

$$C(\Gamma_1) \simeq C(\Gamma_0) \otimes_{\text{Sym}(\mathbb{A} \cup \mathbb{B})} (\text{Sym}(\mathbb{A}|\mathbb{B})/\text{Sym}(\mathbb{A} \cup \mathbb{B}))\{q^{-mn}\}.$$

The natural inclusion map $\text{Sym}(\mathbb{A} \cup \mathbb{B}) \hookrightarrow \text{Sym}(\mathbb{A}|\mathbb{B})$, which is $\text{Sym}(\mathbb{A} \cup \mathbb{B})$ -linear and has grading 0, induces a homogeneous morphism

$$C(\Gamma_0) \xrightarrow{\phi'} C(\Gamma_1) (\simeq C(\Gamma_0) \otimes_{\text{Sym}(\mathbb{A} \cup \mathbb{B})} (\text{Sym}(\mathbb{A}|\mathbb{B})/\text{Sym}(\mathbb{A} \cup \mathbb{B}))\{q^{-mn}\})$$

of quantum degree $-mn$ given by $\phi'(r) = r \otimes 1$.

From Theorem 4.3, there is a unique $\text{Sym}(\mathbb{A} \cup \mathbb{B})$ -linear homogeneous projection $\zeta : \text{Sym}(\mathbb{A}|\mathbb{B}) \rightarrow \text{Sym}(\mathbb{A} \cup \mathbb{B})$ of degree $-2mn$, called the Sylvester operator, satisfying, for $\lambda, \mu \in \Lambda_{m,n}$,

$$\zeta(S_\lambda(\mathbb{A}) \cdot S_\mu(-\mathbb{B})) = \begin{cases} 1 & \text{if } \lambda_j + \mu_{m+1-j} = n \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

The Sylvester operator ζ induces a homogeneous morphism

$$(C(\Gamma_0) \otimes_{\text{Sym}(\mathbb{A} \cup \mathbb{B})} (\text{Sym}(\mathbb{A}|\mathbb{B})/\text{Sym}(\mathbb{A} \cup \mathbb{B}))\{q^{-mn}\}) \simeq C(\Gamma_1) \xrightarrow{\bar{\phi}'} C(\Gamma_0)$$

of quantum \mathbb{Z}_2 degree 0 and degree $-mn$ given by

$$\bar{\phi}'(r \otimes (S_\lambda(\mathbb{A}) \cdot S_\mu(-\mathbb{B}))) = \begin{cases} r & \text{if } \lambda_j + \mu_{m+1-j} = n \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda, \mu \in \Lambda_{m,n}$.

Clearly, $\bar{\phi}'(S_{\lambda_{m,n}}(\mathbb{A}) \cdot \phi'(r)) = r \ \forall r \in C(\Gamma_0)$. So ϕ' and $\bar{\phi}'$ are not homotopic to 0. Thus, $\phi \approx \phi'$ and $\bar{\phi} \approx \bar{\phi}'$. In particular, we have the following lemma.

Lemma 7.11. *Let Γ_0 and Γ_1 be the MOY graphs in Figure 14. Then*

$$\bar{\phi} \circ \mathfrak{m}(S_\lambda(\mathbb{A}) \cdot S_\mu(-\mathbb{B})) \circ \phi \approx \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \lambda_j + \mu_{m+1-j} = n \ \forall j = 1, \dots, m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda, \mu \in \Lambda_{m,n}$ and $\mathfrak{m}(S_\lambda(\mathbb{A}) \cdot S_\mu(-\mathbb{B}))$ is the morphism induced by the multiplication of $S_\lambda(\mathbb{A}) \cdot S_\mu(-\mathbb{B})$.

7.5. Adjoint Koszul matrix factorizations. Let Γ_0 and Γ_1 be the MOY graphs in Figure 15. Khovanov and Rozansky [18] defined morphisms $C(\Gamma_0) \xrightarrow{\chi^0} C(\Gamma_1)$ and $C(\Gamma_1) \xrightarrow{\chi^1} C(\Gamma_0)$, which play an important role in the construction of their link homology and the proof of its invariance. In this and next subsections, we generalize their construction of these χ -morphisms. First we construct morphisms between adjoint Koszul matrix factorizations in this subsection. Then, in next

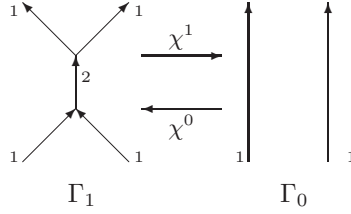


FIGURE 15.

subsection, we apply this construction to matrix factorizations of MOY graphs to define the general χ -morphisms.

Let R be a graded commutative unital \mathbb{C} -algebra. Suppose that, for $i, j = 1, \dots, n$, a_j , b_i and t_{ij} are homogeneous elements of R satisfying $\deg a_j + \deg b_i + \deg t_{ij} = 2N + 2$. Let

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}.$$

Then $M := (A, T^t B)_R$ and $M' := (TA, B)_R$ are both graded Koszul matrix factorizations over R with potential $w = \sum_{i,j=1}^n a_j b_i T_{ij}$. Here, T^t is the transposition of T . We call M and M' adjoint Koszul matrix factorizations and T the relation matrix. Our next objective is to construct a pair of morphisms between M and M' satisfying certain special properties we need. The following is the main result of this subsection.

Proposition 7.12. *Let M and M' be as above. Then there exist morphisms $F : M \rightarrow M'$ and $G : M' \rightarrow M$ satisfying:*

- (i) $\deg_{\mathbb{Z}_2} F = \deg_{\mathbb{Z}_2} G = 0$, $\deg F = 0$ and

$$\deg G = \deg \det(T) = 2n(N + 1) - \sum_{k=1}^n (\deg a_k + \deg b_k).$$

- (ii) $G \circ F = \det(T) \cdot \text{id}_M$ and $F \circ G = \det(T) \cdot \text{id}_{M'}$.

As a special case of Proposition 7.12, we have the following corollary, which was first established in [19, Subsection 2.1].

Corollary 7.13. [19] *Let a, b, t be homogeneous elements of R with $\deg a + \deg b + \deg t = 2N + 2$. Then there exist homogeneous morphisms*

$$\begin{aligned} f &: (a, tb)_R \rightarrow (ta, b)_R, \\ g &: (ta, b)_R \rightarrow (a, tb)_R, \end{aligned}$$

such that

- (i) $\deg_{\mathbb{Z}_2} f = \deg_{\mathbb{Z}_2} g = 0$, $\deg f = 0$ and $\deg g = \deg t$.
(ii) $g \circ f = t \cdot \text{id}_{(a, tb)_R}$ and $f \circ g = t \cdot \text{id}_{(ta, b)_R}$.

Before proving Proposition 7.12, we recall an alternative construction of Koszul matrix factorizations given in [18, Section 2].

Let $R^n = \underbrace{R \oplus \cdots \oplus R}_{n\text{-fold}}$, and $e_i = (0, \dots, 0, \underbrace{1}_{i\text{-th}}, 0, \dots, 0)^t$. The $\{e_1, \dots, e_n\}$ is an R -basis for R^n . Define $T : R^n \rightarrow R^n$ by $T(e_j) = \sum_{i=1}^n T_{ij}e_i$. Let $(R^n)^*$ the dual of R^n over R , $\{e_1^*, \dots, e_n^*\}$ the basis of $(R^n)^*$ dual to $\{e_1, \dots, e_n\}$, and $T^* : (R^n)^* \rightarrow (R^n)^*$ the dual map of T . Then $T^*(e_i^*) = \sum_{j=1}^n T_{ij}e_j^*$. Set

$$\begin{aligned}\alpha &= \sum_{i=1}^n a_i e_i = (e_1, \dots, e_n)A \in R^n, \\ \beta &= \sum_{i=1}^n b_i e_i^* = (e_1^*, \dots, e_n^*)B \in (R^n)^*.\end{aligned}$$

Then $T\alpha = (e_1, \dots, e_n)TA$ and $T^*\beta = (e_1^*, \dots, e_n^*)T^tB$.

From [18, Section 2], we know that $M = (A, T^tB)_R$ is the matrix factorization

$$\bigwedge_{\text{even}} R^n \xrightarrow{\wedge\alpha + \neg T^*\beta} \bigwedge_{\text{odd}} R^n \xrightarrow{\wedge\alpha + \neg T^*\beta} \bigwedge_{\text{even}} R^n,$$

in which, for any $i_1 < \cdots < i_k$, $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is homogeneous with \mathbb{Z}_2 -grading k and quantum grading $k(N+1) - \sum_{l=1}^k \deg a_{i_l}$.

Similarly, $M' = (TA, B)_R$ is the matrix factorization

$$\bigwedge_{\text{even}} R^n \xrightarrow{\wedge T\alpha + \neg\beta} \bigwedge_{\text{odd}} R^n \xrightarrow{\wedge T\alpha + \neg\beta} \bigwedge_{\text{even}} R^n,$$

in which, for any $i_1 < \cdots < i_k$, $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is homogeneous with \mathbb{Z}_2 -grading k and quantum grading $-k(N+1) + \sum_{l=1}^k \deg b_{i_l}$.

Note that T induces an R -algebra endomorphism $T : \bigwedge R^n \rightarrow \bigwedge R^n$ by

$$T(e_{i_1} \wedge \cdots \wedge e_{i_k}) := Te_{i_1} \wedge \cdots \wedge Te_{i_k}.$$

Define R -module map $\mathcal{D} : R^n \oplus (R^n)^* \rightarrow R^n \oplus (R^n)^*$ by $\mathcal{D}(e_i) = e_i^*$ and $\mathcal{D}(e_i^*) = e_i$. Then $\mathcal{D}^2 = \text{id}$. We define $T^t : R^n \rightarrow R^n$ by $T^t = \mathcal{D} \circ T^* \circ \mathcal{D}$. Then the matrix of T^t under the basis $\{e_1, \dots, e_n\}$ is the the transposition of T . T^t induces an R -algebra endomorphism $T^t : \bigwedge R^n \rightarrow \bigwedge R^n$ by

$$T^t(e_{i_1} \wedge \cdots \wedge e_{i_k}) := T^t e_{i_1} \wedge \cdots \wedge T^t e_{i_k}.$$

Next we introduce the Hodge \star -operator. $\star : \bigwedge R^n \rightarrow \bigwedge R^n$ is an R -module map defined so that, for any $i_1 < \cdots < i_k$, $\star(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$, where $(e_{i_1}, \dots, e_{i_k}, e_{j_1}, \dots, e_{j_{n-k}})$ is an even permutation of (e_1, \dots, e_n) .

To simplify the exposition, we use the the following notations in the rest of this subsection. $\mathcal{I}_k := \{I = (i_1, \dots, i_k) | 1 \leq i_1 < \cdots < i_k \leq n\}$. For any $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, \bar{I} is the unique element $\bar{I} = (j_1, \dots, j_{n-k}) \in \mathcal{I}_{n-k}$ such that $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$. We denote by (I, \bar{I}) the parity of the permutation $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ of $(1, \dots, n)$. Also, we write $e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$. Note that $\star e_I = (-1)^{(I, \bar{I})} e_{\bar{I}}$. Moreover, for $I = (i_1, \dots, i_k)$, $L = (l_1, \dots, l_k) \in \mathcal{I}_k$, we denote by T_{LI} the minor matrix

$$T_{LI} = \begin{pmatrix} T_{l_1 i_1} & T_{l_1 i_2} & \cdots & T_{l_1 i_k} \\ T_{l_2 i_1} & T_{l_2 i_2} & \cdots & T_{l_2 i_k} \\ \cdots & \cdots & \cdots & \cdots \\ T_{l_k i_1} & T_{l_k i_2} & \cdots & T_{l_k i_k} \end{pmatrix}.$$

Lemma 7.14. *For any $I = (i_1, \dots, i_k) \in \mathcal{I}_k$,*

$$\star T^t \star T(e_I) = T \star T^t \star (e_I) = (-1)^{k(n-k)} \det(T) \cdot e_I.$$

Proof. We first prove

$$(7.4) \quad \star T^t \star T(e_I) = (-1)^{k(n-k)} \det(T) \cdot e_I.$$

Note that

$$\begin{aligned} T(e_I) &= Te_{i_1} \wedge \cdots \wedge Te_{i_k} \\ &= \left(\sum_{j_1=1}^n T_{j_1 i_1} e_{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_k=1}^n T_{j_k i_k} e_{j_k} \right) \\ &= \sum_{J \in \mathcal{I}_k} \det(T_{JI}) \cdot e_J, \\ \star e_J &= (-1)^{(J, \bar{J})} e_{\bar{J}}, \\ T^t(e_{\bar{J}}) &= \sum_{L \in \mathcal{I}_k} \det(T_{\bar{L} \bar{J}}^t) \cdot e_{\bar{L}} = \sum_{L \in \mathcal{I}_k} \det(T_{\bar{J} \bar{L}}) \cdot e_{\bar{L}}, \\ \star e_{\bar{L}} &= (-1)^{(\bar{L}, L)} e_L. \end{aligned}$$

Also, if we write $J = (j_1, \dots, j_k)$ and $L = (l_1, \dots, l_k)$, then

$$\begin{aligned} (J, \bar{J}) &= \sum_{m=1}^k (j_m - m) = \sum_{m=1}^k j_m - \frac{k(k+1)}{2}, \\ (\bar{L}, L) &= \sum_{m=1}^k (n - k + m - l_m) = k(n - k) + \frac{k(k+1)}{2} - \sum_{m=1}^k l_m. \end{aligned}$$

Using the above equations and the Laplace Formula, we get

$$\begin{aligned} &\star T^t \star T(e_I) \\ &= (-1)^{k(n-k)} \sum_{L \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_k} (-1)^{\sum_{m=1}^k j_m - \sum_{m=1}^k l_m} \det(T_{\bar{J} \bar{L}}) \cdot \det(T_{JI}) \cdot e_L \\ &= (-1)^{k(n-k)} \det(T) \cdot e_I. \end{aligned}$$

Thus, (7.4) is true. In particular, if $T = \text{id}$, then $T^t = \text{id}$ and (7.4) implies that

$$(7.5) \quad \star \star (e_I) = (-1)^{k(n-k)} \cdot e_I.$$

Replace T by T^t in (7.4), we get

$$(7.6) \quad \star T \star T^t(e_I) = (-1)^{k(n-k)} \det(T^t) \cdot e_I = (-1)^{k(n-k)} \det(T) \cdot e_I.$$

Note that (7.4), (7.5) and (7.6) are true for all k and all $I \in \mathcal{I}_k$. So we have that

$$\begin{aligned} T \star T^t \star (e_I) &= (-1)^{k(n-k)} \star \star T \star T^t \star (e_I) \\ &= (-1)^{k(n-k)} \star (\star T \star T^t (\star e_I)) \\ &= (-1)^{k(n-k)} \cdot (-1)^{k(n-k)} \cdot \det(T) \cdot \star \star e_I \\ &= (-1)^{k(n-k)} \det(T) \cdot e_I. \end{aligned}$$

□

Lemma 7.15.

$$(7.7) \quad T \circ (\wedge \alpha) = (\wedge T\alpha) \circ T,$$

$$(7.8) \quad T \circ (\neg T^* \beta) = (\neg \beta) \circ T.$$

Proof. For any $I = (i_1, \dots, i_k) \in \mathcal{I}_k$,

$$\begin{aligned} T \circ (\wedge \alpha)(e_{i_1} \wedge \dots \wedge e_{i_k}) &= T(e_{i_1} \wedge \dots \wedge e_{i_k} \wedge \alpha) \\ &= T(e_{i_1} \wedge \dots \wedge e_{i_k}) \wedge T\alpha \\ &= (\wedge T\alpha) \circ T(e_{i_1} \wedge \dots \wedge e_{i_k}). \end{aligned}$$

So (7.7) is true.

Similarly,

$$\begin{aligned} &T \circ (\neg T^* \beta)(e_{i_1} \wedge \dots \wedge e_{i_k}) \\ &= T\left(\sum_{m=1}^k (-1)^{m-1} \beta(Te_{i_m}) \cdot e_{i_1} \wedge \dots \widehat{e_{i_m}} \dots \wedge e_{i_k}\right) \\ &= \sum_{m=1}^k (-1)^{m-1} \beta(Te_{i_m}) \cdot T(e_{i_1}) \wedge \dots \widehat{T(e_{i_m})} \dots \wedge T(e_{i_k}) \\ &= (\neg \beta)(T(e_{i_1}) \wedge \dots \wedge T(e_{i_k})) \\ &= (\neg \beta) \circ T(e_{i_1} \wedge \dots \wedge e_{i_k}). \end{aligned}$$

So (7.8) is true. □

Lemma 7.16.

$$(7.9) \quad \star \circ (\wedge \alpha) = (\neg \mathcal{D}\alpha) \circ \star,$$

$$(7.10) \quad \star \circ (\neg \beta) = (-1)^{n-1} (\wedge \mathcal{D}\beta) \circ \star.$$

Proof. For any $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, let $\bar{I} = (j_1, \dots, j_{n-k})$. Then

$$\begin{aligned} &\star \circ (\wedge \alpha)(e_{i_1} \wedge \dots \wedge e_{i_k}) \\ &= \star(e_{i_1} \wedge \dots \wedge e_{i_k} \wedge \alpha) \\ &= \star\left(\sum_{m=1}^{n-k} a_{j_m} \cdot e_{i_1} \wedge \dots \wedge e_{i_k} \wedge e_{j_m}\right) \\ &= \sum_{m=1}^{n-k} a_{j_m} \cdot \star(e_{i_1} \wedge \dots \wedge e_{i_k} \wedge e_{j_m}) \\ &= \sum_{m=1}^{n-k} a_{j_m} \cdot (-1)^{(I, \bar{I})+m-1} \cdot e_{j_1} \wedge \dots \widehat{e_{j_m}} \dots \wedge e_{j_{n-k}} \\ &= (\neg \mathcal{D}\alpha) \circ \star(e_{i_1} \wedge \dots \wedge e_{i_k}). \end{aligned}$$

So (7.9) is true.

Similarly,

$$\begin{aligned}
& \star \circ (\neg\beta)(e_{i_1} \wedge \cdots \wedge e_{i_k}) \\
&= \star \left(\sum_{m=1}^k (-1)^{m-1} b_{i_m} \cdot e_{i_1} \wedge \cdots \widehat{e_{i_m}} \cdots \wedge e_{i_k} \right) \\
&= \sum_{m=1}^k (-1)^{m-1} b_{i_m} \cdot \star(e_{i_1} \wedge \cdots \widehat{e_{i_m}} \cdots \wedge e_{i_k}) \\
&= \sum_{m=1}^k (-1)^{m-1} b_{i_m} \cdot (-1)^{(I, \bar{I})+n-m} \cdot e_I \wedge e_{i_m} \\
&= (-1)^{n-1} \star(e_I) \wedge \mathcal{D}\beta \\
&= (-1)^{n-1} (\wedge \mathcal{D}\beta) \circ \star(e_{i_1} \wedge \cdots \wedge e_{i_k}).
\end{aligned}$$

So (7.10) is true. \square

Lemma 7.17.

$$(7.11) \quad \mathcal{D} \circ T^t \circ \mathcal{D} = T^*,$$

$$(7.12) \quad \mathcal{D} \circ T(\alpha) = (T^t)^* \circ \mathcal{D}(\alpha).$$

Proof. Recall that T^t is defined by $T^t = \mathcal{D} \circ T^* \circ \mathcal{D}$ and that $\mathcal{D}^2 = \text{id}$. (7.11) follows immediately. Replace T by T^t in (7.11), we get $\mathcal{D} \circ T \circ \mathcal{D} = (T^t)^*$. Plug $\mathcal{D}(\alpha)$ into this equation, we get (7.12). \square

Lemma 7.18.

$$(7.13) \quad (\star T^t \star) \circ (\wedge T \alpha) = (-1)^{n-1} (\wedge \alpha) \circ (\star T^t \star),$$

$$(7.14) \quad (\star T^t \star) \circ (\neg\beta) = (-1)^{n-1} (\neg T^* \beta) \circ (\star T^t \star).$$

Proof. Note that Lemmas 7.15 through 7.17 are true for any $\alpha \in R^n$, $\beta \in (R^n)^*$ and $T \in \text{Hom}_R(R^n, R^n)$. So

$$\begin{aligned}
(\star T^t \star) \circ (\wedge T \alpha) &= (\star T^t) \circ (\neg \mathcal{D} T \alpha) \circ \star && \text{(by (7.9))} \\
&= (\star T^t) \circ (\neg (T^t)^* \mathcal{D} \alpha) \circ \star && \text{(by (7.12))} \\
&= \star \circ (\neg \mathcal{D} \alpha) \circ (T^t \star) && \text{(by (7.8))} \\
&= (-1)^{n-1} (\wedge \mathcal{D}^2 \alpha) \circ (\star T^t \star) && \text{(by (7.10))} \\
&= (-1)^{n-1} (\wedge \alpha) \circ (\star T^t \star) && \text{since } \mathcal{D}^2 = \text{id}.
\end{aligned}$$

This proves (7.13).

Similarly, we have

$$\begin{aligned}
(\star T^t \star) \circ (\neg\beta) &= (-1)^{n-1} (\star T^t) \circ (\wedge \mathcal{D} \beta) \circ \star && \text{(by (7.10))} \\
&= (-1)^{n-1} \star \circ (\wedge T^t \mathcal{D} \beta) \circ (T^t \star) && \text{(by (7.7))} \\
&= (-1)^{n-1} (\neg \mathcal{D} T^t \mathcal{D} \beta) \circ (\star T^t \star) && \text{(by (7.9))} \\
&= (-1)^{n-1} (\neg T^* \beta) \circ (\star T^t \star) && \text{(by (7.11))}
\end{aligned}$$

This proves (7.14). \square

Now we are ready to prove Proposition 7.12.

Proof of Proposition 7.12. Define $F : M \rightarrow M'$ by $F = T : \bigwedge R^n \rightarrow \bigwedge R^n$. Also, define $G : M' \rightarrow M$ by $G(e_I) = (-1)^{k(n-k)} \star F^t \star(e_I) \forall I = (i_1, \dots, i_k) \in \mathcal{I}_k$. Then Lemmas 7.15 and 7.18 imply that F and G are morphisms of matrix factorizations. Lemma 7.14 implies that $G \circ F = \det(T) \cdot \text{id}_M$ and $F \circ G = \det(T) \cdot \text{id}_{M'}$. It is

easy to see that $\deg_{\mathbb{Z}_2} F = \deg_{\mathbb{Z}_2} G = 0$. It remains to show that F and G are homogeneous with the correct quantum gradings.

For $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, let

$$\begin{aligned} S(I) &= \sum_{m=1}^k i_m, \\ S_a(I) &= \sum_{m=1}^k \deg a_{i_m}, \\ S_b(I) &= \sum_{m=1}^k \deg b_{i_m}. \end{aligned}$$

Recall that, e_I is a homogeneous element of both M and M' . As an element of M , the quantum grading of e_I is $\deg_M e_I = k(N+1) - S_a(I)$. And, as an element of M' , its quantum grading is $\deg_{M'} e_I = S_b(I) - k(N+1)$. It is easy to check that, for $I, J \in \mathcal{I}_k$, $\deg T_{JI}$ is homogeneous with $\deg T_{JI} = 2k(N+1) - S_a(I) - S_b(J)$. So

$$\begin{aligned} \deg_{M'} \det(T_{JI}) e_J &= 2k(N+1) - S_a(I) - S_b(J) + S_b(J) - k(N+1) \\ &= k(N+1) - S_a(I) = \deg_M e_I. \end{aligned}$$

But

$$F(e_I) = T(e_I) = \sum_{J \in \mathcal{I}_k} \det(T_{JI}) e_J.$$

This shows that F is homogeneous with quantum degree 0.

Similarly,

$$\begin{aligned} G(e_I) &= (-1)^{k(n-k)} \star T^t \star (e_I) = \sum_{J \in \mathcal{I}_k} (-1)^{S(I)+S(J)} \det(T_{JI}^t) e_J \\ &= \sum_{J \in \mathcal{I}_k} (-1)^{S(I)+S(J)} \det(T_{\bar{I}\bar{J}}) e_J. \end{aligned}$$

Note that each term $\det(T_{\bar{I}\bar{J}}) e_J$ is homogeneous in M with quantum degree

$$\begin{aligned} &\deg_M \det(T_{\bar{I}\bar{J}}) e_J \\ &= 2(n-k)(N+1) - S_a(\bar{J}) - S_b(\bar{I}) + k(N+1) - S_a(J) \\ &= (2n-k)(N+1) - S_a(1, \dots, n) - (S_b(1, \dots, n) - S_b(I)) \\ &= (2n(N+1) - S_a(1, \dots, n) - S_b(1, \dots, n)) + (S_b(I) - k(N+1)) \\ &= \deg \det(T) + \deg_{M'} e_I. \end{aligned}$$

This shows that G is homogeneous with quantum degree $\deg \det(T)$. \square

Remark 7.19. First, note that Lemma 2.12 and Corollary 2.16 are both special cases of Proposition 7.12. Second, recall that Rasmussen [34] explained that the \mathbb{Z}_2 -grading of a Koszul matrix factorization can be lifted to a \mathbb{Z} -grading. The above construction clearly preserves this \mathbb{Z} -grading.

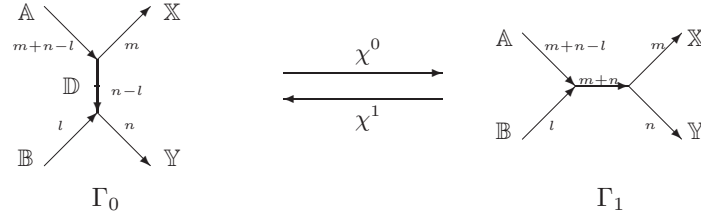


FIGURE 16.

7.6. General χ -morphisms. The following proposition is the main result of this subsection.

Proposition 7.20. *Let Γ_0 and Γ_1 be the MOY graphs in Figure 16, where $1 \leq l \leq n < m+n \leq N$. There exist homogeneous morphisms $\chi^0 : C(\Gamma_0) \rightarrow C(\Gamma_1)$ and $\chi^1 : C(\Gamma_1) \rightarrow C(\Gamma_0)$ satisfying*

- (i) Both χ^0 and χ^1 have \mathbb{Z}_2 -grading 0 and quantum grading ml .
- (ii)

$$\begin{aligned} \chi^1 \circ \chi^0 &\simeq \left(\sum_{\lambda \in \Lambda_{l,m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{X}) S_{\lambda^c}(\mathbb{B}) \right) \cdot \text{id}_{C(\Gamma_0)}, \\ \chi^0 \circ \chi^1 &\simeq \left(\sum_{\lambda \in \Lambda_{l,m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{X}) S_{\lambda^c}(\mathbb{B}) \right) \cdot \text{id}_{C(\Gamma_1)}, \end{aligned}$$

where $\Lambda_{l,m} = \{\mu \mid \mu \leq \lambda_{l,m}\} = \{\mu = (\mu_1 \geq \dots \geq \mu_l) \mid l(\mu) \leq l, \mu_1 \leq m\}$, $\lambda' \in \Lambda_{m,l}$ is the conjugate of λ , and λ^c is the complement of λ in $\Lambda_{l,m}$, i.e., if $\lambda = (\lambda_1 \geq \dots \geq \lambda_l) \in \Lambda_{l,m}$, then $\lambda^c = (m - \lambda_1 \geq \dots \geq m - \lambda_l)$.

Before proving Proposition 7.20, we first simplify $C(\Gamma_0)$ and $C(\Gamma_1)$ and represent their homotopy classes by adjoint Koszul matrix factorizations.

Let $R = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$. Denote by X_i the i -th elementary symmetric polynomial in \mathbb{X} and so on. Recall that

$$C(\Gamma_0) = \begin{pmatrix} * & X_1 + D_1 - A_1 & \\ \dots & \dots & \\ * & \sum_{i=0}^{n-l} X_{k-i} D_i - A_k & \\ \dots & \dots & \\ * & X_m D_{n-l} - A_{m+n-l} & \\ * & Y_1 - D_1 - B_1 & \\ \dots & \dots & \\ * & Y_k - \sum_{i=0}^{n-l} B_{k-i} D_i & \\ \dots & \dots & \\ * & Y_n - B_l D_{n-l} & \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B}|\mathbb{D})} \{q^{-m(n-l)}\}.$$

We exclude \mathbb{D} from the base ring by applying Proposition 2.19 to the rows

$$\begin{pmatrix} * & Y_1 - D_1 - B_1 & \\ \dots & \dots & \\ * & Y_{n-l} - \sum_{i=0}^{n-l} B_{n-l-i} D_i & \end{pmatrix}.$$

This gives us

$$(7.15) \quad C(\Gamma_0) \simeq \begin{pmatrix} * & X_1 + D_1 - A_1 & \\ \cdots & \cdots & \\ * & \sum_{i=0}^{n-l} X_{k-i} D_i - A_k & \\ \cdots & \cdots & \\ * & X_m D_{n-l} - A_{m+n-l} & \\ * & Y_{n-l+1} - \sum_{i=0}^{n-l} B_{n-l+1-i} D_i & \\ \cdots & \cdots & \\ * & Y_{n-l+k} - \sum_{i=0}^{n-l} B_{n-l+k-i} D_i & \\ \cdots & \cdots & \\ * & Y_n - B_l D_{n-l} & \end{pmatrix}_R \{q^{-m(n-l)}\},$$

where

$$D_k = \begin{cases} \sum_{i=0}^k (-1)^i h_i(\mathbb{B}) Y_{k-i} & \text{if } k = 0, 1, \dots, n-l, \\ 0 & \text{otherwise.} \end{cases}$$

Since the above sum will appear repeatedly in this subsection, we set

$$T_k = \begin{cases} \sum_{i=0}^k (-1)^i h_i(\mathbb{B}) Y_{k-i} & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Now consider $Y_{n-l+k} - \sum_{i=0}^{n-l} B_{n-l+k-i} D_i$. If $k = 1$, using equation (4.1), we get

$$\begin{aligned} Y_{n-l+1} - \sum_{i=0}^{n-l} B_{n-l+1-i} D_i &= Y_{n-l+1} - \sum_{i=0}^{n-l} B_{n-l+1-i} \sum_{j=0}^i (-1)^{i-j} h_{i-j}(\mathbb{B}) Y_j \\ &= Y_{n-l+1} - \sum_{j=0}^{n-l} Y_j \sum_{i=j}^{n-l} (-1)^{i-j} h_{i-j}(\mathbb{B}) B_{n-l+1-i} \\ &= Y_{n-l+1} - \sum_{j=0}^{n-l} Y_j \sum_{i=0}^{n-l-j} (-1)^i h_i(\mathbb{B}) B_{n-l+1-j-i} \\ &= Y_{n-l+1} + \sum_{j=0}^{n-l} (-1)^{n-l+1-j} Y_j h_{n-l+1-j}(\mathbb{B}) \\ &= T_{n-l+1}. \end{aligned}$$

If $k > 1$, then

$$\begin{aligned}
& Y_{n-l+k} - \sum_{i=0}^{n-l} B_{n-l+k-i} D_i \\
= & Y_{n-l+k} - \sum_{i=0}^{n-l} B_{n-l+k-i} \sum_{j=0}^i (-1)^{i-j} h_{i-j}(\mathbb{B}) Y_j \\
= & Y_{n-l+k} - \sum_{j=0}^{n-l} Y_j \sum_{i=j}^{n-l} (-1)^{i-j} h_{i-j}(\mathbb{B}) B_{n-l+k-i} \\
= & Y_{n-l+k} - \sum_{j=0}^{n-l} Y_j \sum_{i=0}^{n-l-j} (-1)^i h_i(\mathbb{B}) B_{n-l+k-j-i} \\
= & Y_{n-l+k} + \sum_{j=0}^{n-l} Y_j \sum_{i=n-l-j+1}^{n-l-j+k} (-1)^i h_i(\mathbb{B}) B_{n-l+k-j-i} \\
= & Y_{n-l+k} + \sum_{j=0}^{n-l} Y_j \sum_{i=0}^{k-1} (-1)^{n-l+k-j-i} h_{n-l+k-j-i}(\mathbb{B}) B_i \\
= & Y_{n-l+k} + \sum_{i=0}^{k-1} B_i \sum_{j=0}^{n-l} Y_j (-1)^{n-l+k-j-i} h_{n-l+k-j-i}(\mathbb{B}) \\
= & Y_{n-l+k} + \sum_{i=0}^{k-1} B_i (T_{n-l+k-i} - \sum_{j=n-l+1}^{n-l+k-i} Y_j (-1)^{n-l+k-j-i} h_{n-l+k-j-i}(\mathbb{B})) \\
= & Y_{n-l+k} + \sum_{i=0}^{k-1} B_i T_{n-l+k-i} - \sum_{i=0}^{k-1} B_i \sum_{j=n-l+1}^{n-l+k-i} Y_j (-1)^{n-l+k-j-i} h_{n-l+k-j-i}(\mathbb{B}).
\end{aligned}$$

But, by equation (4.1),

$$\begin{aligned}
& \sum_{i=0}^{k-1} B_i \sum_{j=n-l+1}^{n-l+k-i} Y_j (-1)^{n-l+k-j-i} h_{n-l+k-j-i}(\mathbb{B}) \\
= & \sum_{i=0}^{k-1} B_i \sum_{j=0}^{k-1-i} Y_{j+n-l+1} (-1)^{k-1-j-i} h_{k-1-j-i}(\mathbb{B}) \\
= & \sum_{j=0}^{k-1} Y_{j+n-l+1} \sum_{i=0}^{k-1-j} (-1)^{k-1-j-i} h_{k-1-j-i}(\mathbb{B}) B_i \\
= & Y_{k+n-l}.
\end{aligned}$$

So

$$Y_{n-l+k} - \sum_{i=0}^{n-l} B_{n-l+k-i} D_i = Y_{n-l+k} + \sum_{i=0}^{k-1} B_i T_{n-l+k-i} - Y_{k+n-l} = \sum_{i=0}^{k-1} B_i T_{n-l+k-i}$$

and, therefore,

$$Y_{n-l+k} - \sum_{i=0}^{n-l} B_{n-l+k-i} D_i - \sum_{i=1}^{k-1} B_i T_{n-l+k-i} = T_{n-l+k}.$$

Thus, we can apply Corollary 2.16 successively to the right hand side of (7.15) to get

$$(7.16) \quad C(\Gamma_0) \simeq \left(\begin{array}{cc} * & X_1 + D_1 - A_1 \\ \cdots & \cdots \\ * & \sum_{i=0}^{n-l} X_{k-i} D_i - A_k \\ \cdots & \cdots \\ * & X_m D_{n-l} - A_{m+n-l} \\ * & T_{n-l+1} \\ \cdots & \cdots \\ * & T_n \end{array} \right)_R \{q^{-m(n-l)}\}.$$

Lemma 7.21. *If $k > n$, then $T_k = -\sum_{j=1}^l B_j T_{k-j}$.*

Proof. For $k > n$,

$$\begin{aligned} T_k &= \sum_{i=0}^k (-1)^i h_i(\mathbb{B}) Y_{k-i} = \sum_{i=0}^n (-1)^{k-i} h_{k-i}(\mathbb{B}) Y_i \\ (\text{by (4.1), note that } k > n) \quad &= -\sum_{i=0}^n (-1)^{k-i} Y_i \sum_{j=1}^l (-1)^j B_j h_{k-i-j}(\mathbb{B}) \\ &= -\sum_{j=1}^l B_j \sum_{i=0}^n (-1)^{k-i-j} Y_i h_{k-i-j}(\mathbb{B}) \\ &= -\sum_{j=1}^l B_j T_{k-j}. \end{aligned}$$

□

Lemma 7.22. *For any $k \geq 0$, define $W_k = \sum_{i=0}^k T_i X_{k-i}$. Then*

$$(7.17) \quad C(\Gamma_0) \simeq \left(\begin{array}{cc} * & W_1 - A_1 \\ \cdots & \cdots \\ * & W_k - A_k \\ \cdots & \cdots \\ * & W_{m+n-l} - A_{m+n-l} \\ * & T_{n-l+1} \\ \cdots & \cdots \\ * & T_n \end{array} \right)_R \{q^{-m(n-l)}\}.$$

Proof. Consider $\sum_{i=0}^{n-l} X_{k-i} D_i - A_k$. If $k \leq n-l$, then

$$\sum_{i=0}^{n-l} X_{k-i} D_i - A_k = \sum_{i=0}^k X_{k-i} T_i - A_k = W_k - A_k.$$

So the row $(*, \sum_{i=0}^{n-l} X_{k-i} D_i - A_k)$ in (7.16) is already $(*, W_k - A_k)$.

If $k > n - l$, then

$$\sum_{i=0}^{n-l} X_{k-i} D_i - A_k = \sum_{i=0}^{n-l} X_{k-i} T_i - A_k = W_k - A_k - \sum_{i=n-l+1}^k X_{k-i} T_i.$$

By Lemma 7.21, if $i \geq n - l + 1$, then T_i can be expressed as a combination of T_{n-l+1}, \dots, T_n . So we can apply Corollary 2.16 to the row $(*, \sum_{i=0}^{n-l} X_{k-i} D_i - A_k)$ and the bottom l rows in (7.16) to change the former into $(*, W_k - A_k)$. \square

Now consider Γ_1 in Figure 16. Recall that

$$(7.18) \quad C(\Gamma_1) \simeq \left(\begin{array}{cc} * & X_1 + Y_1 - A_1 - B_1 \\ \cdots & \cdots \\ * & \sum_{i=0}^k X_i Y_{k-i} - \sum_{i=0}^k A_i B_{k-i} \\ \cdots & \cdots \\ * & X_m Y_n - A_{m+n-l} B_l \end{array} \right)_R \{q^{-mn}\}.$$

Lemma 7.23.

$$(7.19) \quad C(\Gamma_1) \simeq \left(\begin{array}{cc} * & W_1 - A_1 \\ \cdots & \cdots \\ * & W_k - A_k \\ \cdots & \cdots \\ * & W_{m+n-l} - A_{m+n-l} \\ * & W_{m+n-l+1} \\ \cdots & \cdots \\ * & W_{m+n} \end{array} \right)_R \{q^{-mn}\},$$

where, as in Lemma 7.22, $W_k = \sum_{i=0}^k T_i X_{k-i}$.

Proof. We have

$$\begin{aligned} \sum_{j=0}^k (-1)^{k-j} h_{k-j}(\mathbb{B}) \sum_{i=0}^j X_i Y_{j-i} &= \sum_{i=0}^k X_i \sum_{j=i}^k (-1)^{k-j} h_{k-j}(\mathbb{B}) Y_{j-i} \\ &= \sum_{i=0}^k X_i \sum_{j=0}^{k-i} (-1)^{k-i-j} h_{k-i-j}(\mathbb{B}) Y_j \\ &= \sum_{i=0}^k X_i T_{k-i} = W_k \end{aligned}$$

and, by (4.1),

$$\begin{aligned} \sum_{j=0}^k (-1)^{k-j} h_{k-j}(\mathbb{B}) \sum_{i=0}^j A_i B_{j-i} &= \sum_{i=0}^k A_i \sum_{j=i}^k (-1)^{k-j} h_{k-j}(\mathbb{B}) B_{j-i} \\ &= \sum_{i=0}^k A_i \sum_{j=0}^{k-i} (-1)^{k-i-j} h_{k-i-j}(\mathbb{B}) B_j \\ &= A_k. \end{aligned}$$

Thus,

$$\begin{aligned}
& \sum_{j=1}^k (-1)^{k-j} h_{k-j}(\mathbb{B}) \left(\sum_{i=0}^j X_i Y_{j-i} - \sum_{i=0}^j A_i B_{j-i} \right) \\
&= \sum_{j=0}^k (-1)^{k-j} h_{k-j}(\mathbb{B}) \left(\sum_{i=0}^j X_i Y_{j-i} - \sum_{i=0}^j A_i B_{j-i} \right) \\
&= W_k - A_k.
\end{aligned}$$

This implies that we can get (7.19) by successively apply Corollary 2.16 to the right hand side of (7.18). \square

By the definition of W_k , we know that

$$\begin{pmatrix} W_{m+n} \\ W_{m+n-1} \\ \dots \\ W_{m+n-l+1} \end{pmatrix} = \Omega \begin{pmatrix} T_{m+n} \\ T_{m+n-1} \\ \dots \\ T_{n-l+1} \end{pmatrix},$$

where Ω is an $l \times (m+l)$ matrix whose (i, j) -th entry is X_{j-i} . By Lemma 7.21, we have, for $k \geq 1$,

$$\begin{pmatrix} T_{n+k} \\ T_{n+k-1} \\ \dots \\ T_{n-l+1} \end{pmatrix} = \Theta_k \begin{pmatrix} T_{n+k-1} \\ T_{n+k-2} \\ \dots \\ T_{n-l+1} \end{pmatrix},$$

where Θ_k is the $(k+l) \times (k+l-1)$ matrix whose first row is $(-B_1, -B_2, \dots, -B_l, 0, \dots, 0)$ and whose next $(k+l-1)$ rows form the $(k+l-1) \times (k+l-1)$ identity matrix Id_{k+l-1} . Define $\Theta = \Theta_m \Theta_{m-1} \dots \Theta_2 \Theta_1$. Then

$$\begin{pmatrix} W_{m+n} \\ W_{m+n-1} \\ \dots \\ W_{m+n-l+1} \end{pmatrix} = \Omega \Theta \begin{pmatrix} T_n \\ T_{n-1} \\ \dots \\ T_{n-l+1} \end{pmatrix},$$

where $\Omega \Theta$ is clearly an $l \times l$ matrix. So

$$(7.20) \quad \begin{pmatrix} W_1 - A_1 \\ \dots \\ W_{m+n-l} - A_{m+n-l} \\ W_{m+n} \\ \dots \\ W_{m+n-l+1} \end{pmatrix} = \begin{pmatrix} \text{Id}_{m+n-l} & 0 \\ 0 & \Omega \Theta \end{pmatrix} \begin{pmatrix} W_1 - A_1 \\ \dots \\ W_{m+n-l} - A_{m+n-l} \\ T_n \\ \dots \\ T_{n-l+1} \end{pmatrix}.$$

Lemma 7.24. *The homotopy classes of $C(\Gamma_0)\{q^{m(n-l)}\}$ and $C(\Gamma_1)\{q^{mn}\}$ can be represented by adjoint Koszul matrix factorizations with relation matrix*

$$\begin{pmatrix} \text{Id}_{m+n-l} & 0 \\ 0 & \Omega \Theta \end{pmatrix}^t = \begin{pmatrix} I_{m+n-l} & 0 \\ 0 & \Theta^t \Omega^t \end{pmatrix}.$$

Proof. This lemma follows from (7.20), Lemmas 7.22, 7.23 and Remark 5.4. \square

The morphisms χ^0 and χ^1 in Proposition 7.20 are constructed by applying Proposition 7.12 to this pair of adjoint matrix factorizations in Lemma 7.24. To prove

that χ^0 and χ^1 satisfy all the requirements in Proposition 7.20, all we need to do is to compute

$$\det \begin{pmatrix} \text{Id}_{m+n-l} & 0 \\ 0 & \Omega\Theta \end{pmatrix}^t = \det(\Omega\Theta).$$

For this purpose, we introduce some results about Schur polynomials associated to hook partitions.

For $i, j \geq 0$, let $L_{i,j} = (i+1 \geq \underbrace{1 \geq \dots \geq 1}_j)$, the (i, j) -hook partition. Note that

$L'_{i,j} = L_{j,i}$. So

$$(7.21) \quad S_{L_{i,j}}(\mathbb{B}) = \begin{vmatrix} h_{i+1}(\mathbb{B}) & h_{i+2}(\mathbb{B}) & h_{i+3}(\mathbb{B}) & \cdots & h_{i+j}(\mathbb{B}) & h_{i+j+1}(\mathbb{B}) \\ 1 & h_1(\mathbb{B}) & h_2(\mathbb{B}) & \cdots & h_{j-1}(\mathbb{B}) & h_j(\mathbb{B}) \\ 0 & 1 & h_1(\mathbb{B}) & \cdots & h_{j-2}(\mathbb{B}) & h_{j-1}(\mathbb{B}) \\ 0 & 0 & 1 & \cdots & h_{j-3}(\mathbb{B}) & h_{j-2}(\mathbb{B}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & h_1(\mathbb{B}) \end{vmatrix},$$

$$(7.22) \quad S_{L_{i,j}}(\mathbb{B}) = \begin{vmatrix} B_{j+1} & B_{j+2} & B_{j+3} & \cdots & B_{j+i} & B_{j+i+1} \\ 1 & B_1 & B_2 & \cdots & B_{i-1} & B_i \\ 0 & 1 & B_1 & \cdots & B_{i-3} & B_{i-1} \\ 0 & 0 & 1 & \cdots & B_{i-3} & B_{i-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & B_1 \end{vmatrix}.$$

Using (7.21) and (7.22), we can extend the definition of $S_{L_{i,j}}(\mathbb{B})$ to allow one of i, j to be negative.

(i) If $j \geq 0$, then

$$S_{L_{i,j}}(\mathbb{B}) = \begin{cases} S_{L_{i,j}}(\mathbb{B}) \text{ as in (7.21)} & \text{if } i \geq 0, \\ (-1)^j & \text{if } i = -j - 1, \\ 0 & \text{if } i < 0 \text{ and } i \neq -j - 1. \end{cases}$$

(ii) If $i \geq 0$, then

$$S_{L_{i,j}}(\mathbb{B}) = \begin{cases} S_{L_{i,j}}(\mathbb{B}) \text{ as in (7.22)} & \text{if } j \geq 0, \\ (-1)^i & \text{if } j = -i - 1, \\ 0 & \text{if } j < 0 \text{ and } j \neq -i - 1. \end{cases}$$

Lemma 7.25. Define $\tau_{i,j} = (-1)^{i+1} S_{L_{i,j}}(\mathbb{B})$. Then, for $i, j \geq 0$,

$$(7.23) \quad B_{i+j+1} = - \sum_{k=0}^i B_k \tau_{i-k,j},$$

$$(7.24) \quad (-1)^{i+j+1} h_{i+j+1}(\mathbb{B}) = \sum_{k=0}^j (-1)^k h_k(\mathbb{B}) \tau_{i,j-k}.$$

Proof. Let us prove (7.23) first. Note that, for any $i \geq 0$,

$$h_{i+1}(\mathbb{B}) = \sum_{k=1}^{\infty} (-1)^{k+1} B_k h_{i+1-k}(\mathbb{B}),$$

where the right hand side is in fact a finite sum. Apply this equation to every entry in the first row of (7.21), we get

$$\begin{aligned} S_{L_{i,j}}(\mathbb{B}) &= \sum_{k=1}^{\infty} (-1)^{k+1} B_k S_{L_{i-k,j}}(\mathbb{B}) \\ \text{(by (i) above)} &= \left(\sum_{k=1}^i (-1)^{k+1} B_k S_{L_{i-k,j}}(\mathbb{B}) \right) + (-1)^i B_{i+j+1}. \end{aligned}$$

Equation (7.23) follows from this and the definition of $\tau_{i,j}$.

Now we prove (7.24). For any $j \geq 0$,

$$B_{j+1} = \sum_{k=1}^{\infty} (-1)^{k+1} h_k(\mathbb{B}) B_{j+1-k},$$

where the right hand side is again a finite sum. Apply this equation to every entry in the first row of (7.22), we get

$$\begin{aligned} S_{L_{i,j}}(\mathbb{B}) &= \sum_{k=1}^{\infty} (-1)^{k+1} h_k(\mathbb{B}) S_{L_{i,j-k}}(\mathbb{B}) \\ \text{(by (ii) above)} &= \left(\sum_{k=1}^j (-1)^{k+1} h_k(\mathbb{B}) S_{L_{i,j-k}}(\mathbb{B}) \right) + (-1)^j h_{i+j+1}(\mathbb{B}). \end{aligned}$$

Equation (7.24) follows from this and the definition of $\tau_{i,j}$. □

Lemma 7.26.

$$\Theta = (\tau_{m-i,j-1})_{(l+m) \times l} = \begin{pmatrix} \tau_{m-1,0} & \tau_{m-1,1} & \cdots & \tau_{m-1,l-1} \\ \tau_{m-2,0} & \tau_{m-2,1} & \cdots & \tau_{m-2,l-1} \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{0,0} & \tau_{0,1} & \cdots & \tau_{0,l-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Proof. Note that $\tau_{0,j-1} = -B_j$. Recall that

$$\Theta_1 = \begin{pmatrix} -B_1 & -B_2 & \cdots & B_l \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} & \cdots & \tau_{0,l-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$\Theta_k = \begin{pmatrix} \Theta_1 & 0 \\ 0 & \text{Id}_{k-1} \end{pmatrix}.$$

Using equation (7.23) in Lemma 7.25, it is easy to prove by induction that

$$\Theta_k \Theta_{k-1} \cdots \Theta_1 = (\tau_{k-i,j-1})_{(l+k) \times l} = \begin{pmatrix} \tau_{k-1,0} & \tau_{k-1,1} & \cdots & \tau_{k-1,l-1} \\ \tau_{k-2,0} & \tau_{k-2,1} & \cdots & \tau_{k-2,l-1} \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{0,0} & \tau_{0,1} & \cdots & \tau_{0,l-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

But $\Theta = \Theta_m \Theta_{m-1} \cdots \Theta_1$. So the lemma is the $k = m$ case of the above equation. \square

Define

- $u_{i,j} = (-1)^{j-i} h_{j-i}(\mathbb{B})$ for $1 \leq i, j \leq l$,
- $v_{i,j} = (-1)^{j+m-i} h_{j+m-i}(\mathbb{B})$ for $1 \leq i \leq l+m$ and $1 \leq j \leq l$.

Let

$$U = (u_{i,j})_{l \times l} = \begin{pmatrix} 1 & -h_1(\mathbb{B}) & \cdots & (-1)^{l-2} h_{l-2}(\mathbb{B}) & (-1)^{l-1} h_{l-1}(\mathbb{B}) \\ 0 & 1 & \cdots & (-1)^{l-3} h_{l-3}(\mathbb{B}) & (-1)^{l-2} h_{l-2}(\mathbb{B}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -h_1(\mathbb{B}) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$V = (v_{i,j})_{(l+m) \times l} = \begin{pmatrix} (-1)^m h_m(\mathbb{B}) & (-1)^{m+1} h_{m+1}(\mathbb{B}) & \cdots & (-1)^{m+l-2} h_{m+l-2}(\mathbb{B}) & (-1)^{m+l-1} h_{m+l-1}(\mathbb{B}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -h_1(\mathbb{B}) & h_2(\mathbb{B}) & \cdots & (-1)^{l-1} h_{l-1}(\mathbb{B}) & (-1)^l h_l(\mathbb{B}) \\ 1 & -h_1(\mathbb{B}) & \cdots & (-1)^{l-2} h_{l-2}(\mathbb{B}) & (-1)^{l-1} h_{l-1}(\mathbb{B}) \\ 0 & 1 & \cdots & (-1)^{l-3} h_{l-3}(\mathbb{B}) & (-1)^{l-2} h_{l-2}(\mathbb{B}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -h_1(\mathbb{B}) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

Lemma 7.27. $\Theta U = V$.

Proof. This follows from Lemma 7.26 and equation (7.24) in Lemma 7.25. \square

Lemma 7.28.

$$\det \begin{pmatrix} \text{Id}_{m+n-l} & 0 \\ 0 & \Omega \Theta \end{pmatrix}^t = \det(\Omega \Theta) = (-1)^{ml} \sum_{\lambda \in \Lambda_{l,m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{X}) S_{\lambda^c}(\mathbb{B}).$$

Proof. Note that $\det U = 1$. So by Lemma 7.27, $\det(\Omega \Theta) = \det(\Omega \Theta U) = \det(\Omega V)$.

Let

$$\mathcal{I} := \{I = (i_1, \dots, i_l) | 1 \leq i_1 < \cdots < i_l \leq l+m\}.$$

For any $I = (i_1, \dots, i_l) \in \mathcal{I}$, define

- Ω_I to be the $l \times l$ minor matrix of Ω consisting of the i_1, \dots, i_l -th columns of Ω ,
- V_I to be the $l \times l$ minor matrix of V consisting of the i_1, \dots, i_l -th rows of V .

Then, by the Binet-Cauchy Theorem,

$$(7.25) \quad \det(\Omega\Theta) = \det(\Omega V) = \sum_{I \in \mathcal{I}} \det \Omega_I \cdot \det V_I.$$

Recall that $\Lambda_{l,m} = \{\mu \mid \mu \leq \lambda_{l,m}\} = \{\mu = (\mu_1 \geq \dots \geq \mu_l) \mid l(\mu) \leq l, \mu_1 \leq m\}$. Note that there is a one-to-one correspondence $j : \mathcal{I} \rightarrow \Lambda_{l,m}$ given by

$$j(I) = (i_l - l \geq i_{l-1} - l + 1 \geq \dots \geq i_1 - 1)$$

for any $I = (i_1, \dots, i_l) \in \mathcal{I}$. The inverse of j is given by

$$j^{-1}(\lambda) = (\lambda_l + 1, \lambda_{l-1} + 2, \dots, \lambda_1 + l)$$

for any $\lambda = (\lambda_1 \geq \dots \geq \lambda_l) \in \Lambda_{l,m}$.

For any $I = (i_1, \dots, i_l) \in \mathcal{I}$,

$$\begin{aligned} \det \Omega_I &= \begin{vmatrix} X_{i_1-1} & X_{i_2-1} & \dots & X_{i_{l-1}-1} & X_{i_l-1} \\ X_{i_1-2} & X_{i_2-2} & \dots & X_{i_{l-1}-2} & X_{i_l-2} \\ \dots & \dots & \dots & \dots & \dots \\ X_{i_1-l+1} & X_{i_2-l+1} & \dots & X_{i_{l-1}-l+1} & X_{i_l-l+1} \\ X_{i_1-l} & X_{i_2-l} & \dots & X_{i_{l-1}-l} & X_{i_l-l} \end{vmatrix} \\ &= \begin{vmatrix} X_{i_l-l} & X_{i_l-l+1} & \dots & X_{i_l-2} & X_{i_l-1} \\ X_{i_{l-1}-l} & X_{i_{l-1}-l+1} & \dots & X_{i_{l-1}-2} & X_{i_{l-1}-1} \\ \dots & \dots & \dots & \dots & \dots \\ X_{i_2-l} & X_{i_2-l+1} & \dots & X_{i_2-2} & X_{i_2-1} \\ X_{i_1-l} & X_{i_1-l+1} & \dots & X_{i_1-2} & X_{i_1-1} \end{vmatrix} \\ &= S_{j(I)'}(\mathbb{X}). \end{aligned}$$

To shorten the equation, we let $h_i = h_i(\mathbb{B})$ and $\tilde{h}_i = (-1)^i h_i(\mathbb{B})$. Then, for any $I = (i_1, \dots, i_l) \in \mathcal{I}$,

$$\begin{aligned} &\det V_I \\ &= \begin{vmatrix} \tilde{h}_{m+1-i_1} & \tilde{h}_{m+2-i_1} & \dots & \tilde{h}_{m+l-1-i_1} & \tilde{h}_{m+l-i_1} \\ \tilde{h}_{m+1-i_2} & \tilde{h}_{m+2-i_2} & \dots & \tilde{h}_{m+l-1-i_2} & \tilde{h}_{m+l-i_2} \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{h}_{m+1-i_{l-1}} & \tilde{h}_{m+2-i_{l-1}} & \dots & \tilde{h}_{m+l-1-i_{l-1}} & \tilde{h}_{m+l-i_{l-1}} \\ \tilde{h}_{m+1-i_l} & \tilde{h}_{m+2-i_l} & \dots & \tilde{h}_{m+l-1-i_l} & \tilde{h}_{m+l-i_l} \end{vmatrix} \\ &= (-1)^{ml + \frac{l(l+1)}{2} - \sum_{k=1}^l i_k} \begin{vmatrix} h_{m+1-i_1} & h_{m+2-i_1} & \dots & h_{m+l-1-i_1} & h_{m+l-i_1} \\ h_{m+1-i_2} & h_{m+2-i_2} & \dots & h_{m+l-1-i_2} & h_{m+l-i_2} \\ \dots & \dots & \dots & \dots & \dots \\ h_{m+1-i_{l-1}} & h_{m+2-i_{l-1}} & \dots & h_{m+l-1-i_{l-1}} & h_{m+l-i_{l-1}} \\ h_{m+1-i_l} & h_{m+2-i_l} & \dots & h_{m+l-1-i_l} & h_{m+l-i_l} \end{vmatrix} \\ &= (-1)^{ml + |j(I)|} S_{j(I)'c}(\mathbb{B}). \end{aligned}$$

So (7.25) gives that

$$\begin{aligned} \det(\Omega\Theta) &= (-1)^{ml} \sum_{I \in \mathcal{I}} (-1)^{|j(I)|} S_{j(I)'c}(\mathbb{B}) S_{j(I)}(\mathbb{B}) \\ &= (-1)^{ml} \sum_{\lambda \in \Lambda_{l,m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{B}) S_{\lambda^c}(\mathbb{B}). \end{aligned}$$

□

Proof of Proposition 7.20. Proposition 7.20 now follows easily from Proposition 7.12, Lemma 7.24 and Lemma 7.28. \square

Proposition 7.29. *Let $\Gamma_0, \Gamma_1, \chi^0$ and χ^1 be as in Proposition 7.20. Then χ^0 and χ^1 are homotopically non-trivial. Moreover, up to homotopy and scaling, χ^0 (resp. χ^1) is the unique homotopically non-trivial homogeneous morphism of quantum degree ml from $C(\Gamma_0)$ to $C(\Gamma_1)$ (resp. from $C(\Gamma_1)$ to $C(\Gamma_0)$).*

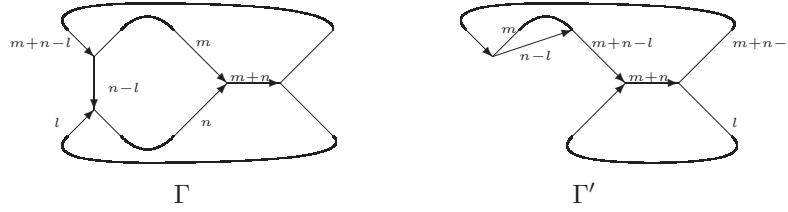


FIGURE 17.

Proof. Using Lemmas 7.22 and 7.23, one can check that, as graded vector spaces,

$$\begin{aligned} \text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0)) &\cong H(\Gamma) \langle m+n \rangle \{q^{(m+n)(N-m-n)+mn+ml+nl-l^2}\}, \\ \text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1)) &\cong H(\bar{\Gamma}) \langle m+n \rangle \{q^{(m+n)(N-m-n)+mn+ml+nl-l^2}\}, \end{aligned}$$

where Γ is the MOY graph in Figure 17, and $\bar{\Gamma}$ is Γ with orientation reversed. By Corollary 5.13, we have $H(\Gamma) \cong H(\Gamma')$. Then, by Decomposition (II) (Theorem 5.14) and Corollary 6.1, we have

$$H(\Gamma) \cong H(\Gamma') \cong C(\emptyset) \langle m+n \rangle \left\{ \begin{bmatrix} m+n-l \\ m \end{bmatrix} \begin{bmatrix} m+n \\ l \end{bmatrix} \begin{bmatrix} N \\ m+n \end{bmatrix} \right\}.$$

And, similarly,

$$H(\bar{\Gamma}) \cong C(\emptyset) \langle m+n \rangle \left\{ \begin{bmatrix} m+n-l \\ m \end{bmatrix} \begin{bmatrix} m+n \\ l \end{bmatrix} \begin{bmatrix} N \\ m+n \end{bmatrix} \right\}.$$

Thus, as graded vector spaces,

$$\begin{aligned} &\text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0)) \\ &\cong \text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1)) \\ &\cong C(\emptyset) \left\{ \begin{bmatrix} m+n-l \\ m \end{bmatrix} \begin{bmatrix} m+n \\ l \end{bmatrix} \begin{bmatrix} N \\ m+n \end{bmatrix} q^{(m+n)(N-m-n)+mn+ml+nl-l^2} \right\}. \end{aligned}$$

In particular, the lowest non-vanishing quantum grading of the above spaces is ml , and the subspaces of these spaces of homogeneous elements of quantum degree ml are 1-dimensional. So, to prove the proposition, we only need to show that χ^0 and χ^1 are homotopically non-trivial. To prove this, we use the diagram in Figure 18.

Consider the morphisms in Figure 18, where ϕ_1 and $\bar{\phi}_1$ (resp. ϕ_2 and $\bar{\phi}_2$) are induced by the edge splitting and merging of the upper (resp. lower) bubble, and χ^0 and χ^1 are the morphisms from Proposition 7.20. Let us compute the composition

$$(\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda_{l,n-l}}(-\mathbb{A})) \circ \chi^0 \circ \chi^1 \circ (\phi_1 \otimes \phi_2),$$

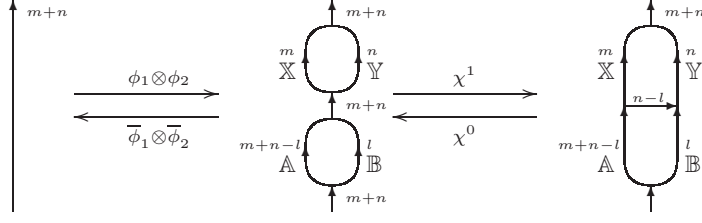


FIGURE 18.

where $\mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda_{l,n-l}}(-\mathbb{A}))$ is the morphism induced by multiplication by $S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda_{l,n-l}}(-\mathbb{A})$. By Proposition 7.20, we have

$$\begin{aligned}
& (\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda_{l,n-l}}(-\mathbb{A})) \circ \chi^0 \circ \chi^1 \circ (\phi_1 \otimes \phi_2) \\
& \simeq (\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda_{l,n-l}}(-\mathbb{A}) \cdot (\sum_{\lambda \in \Lambda_{l,m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{X}) S_{\lambda^c}(\mathbb{B}))) \circ (\phi_1 \otimes \phi_2) \\
& = \sum_{\lambda \in \Lambda_{l,m}} (-1)^{|\lambda|} (\bar{\phi}_1 \circ \mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda'}(\mathbb{X})) \circ \phi_1) \otimes (\bar{\phi}_2 \circ \mathbf{m}(S_{\lambda_{l,n-l}}(-\mathbb{A}) \cdot S_{\lambda^c}(\mathbb{B})) \circ \phi_2).
\end{aligned}$$

But, by Lemma 7.11, we have that, for $\lambda \in \Lambda_{l,m}$,

$$\begin{aligned}
\bar{\phi}_1 \circ \mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda'}(\mathbb{X})) \circ \phi_1 & \approx \begin{cases} \text{id} & \text{if } \lambda = (0 \geq \dots \geq 0), \\ 0 & \text{if } \lambda \neq (0 \geq \dots \geq 0), \end{cases} \\
\bar{\phi}_2 \circ \mathbf{m}(S_{\lambda_{l,n-l}}(-\mathbb{A}) \cdot S_{\lambda^c}(\mathbb{B})) \circ \phi_2 & \approx \begin{cases} \text{id} & \text{if } \lambda = (0 \geq \dots \geq 0), \\ 0 & \text{if } \lambda \neq (0 \geq \dots \geq 0). \end{cases}
\end{aligned}$$

So,

$$(\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \mathbf{m}(S_{\lambda_{m,n}}(-\mathbb{Y}) \cdot S_{\lambda_{l,n-l}}(-\mathbb{A})) \circ \chi^0 \circ \chi^1 \circ (\phi_1 \otimes \phi_2) \approx \text{id},$$

which implies that χ^0 and χ^1 are not homotopic to 0. \square

7.7. Adding and removing a loop. Using the χ -morphisms and the morphisms associated to circle creation and annihilation, one can construction morphisms associated to adding and removing loops.

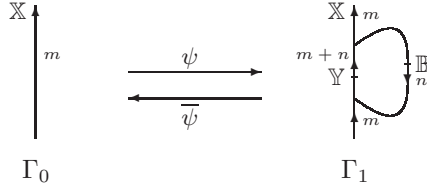


FIGURE 19.

Lemma 7.30. *Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 19. Then, as bigraded vector spaces over \mathbb{C} ,*

$$\text{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1)) \cong \text{Hom}_{HMF}(C(\Gamma_1), C(\Gamma_0)) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \begin{bmatrix} N-m \\ n \end{bmatrix} q^{m(N-m)} \right\} \langle n \rangle.$$

In particular, the subspaces of these spaces of homogeneous elements of quantum degree $-n(N-n) + mn$ is 1-dimensional.

Proof. By Theorem 5.16, we have that $C(\Gamma_1) \simeq C(\Gamma_0)\{ \begin{bmatrix} N-m \\ n \end{bmatrix} \} \langle n \rangle$. So

$$\mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma_0), C(\Gamma_1)) \cong \mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma_0), C(\Gamma_0))\{ \begin{bmatrix} N-m \\ n \end{bmatrix} \} \langle n \rangle \cong \mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma_1), C(\Gamma_0)).$$

Denote by \bigcirc_m a circle colored by m . Then, from the proof of Lemma 6.5, we have

$$\mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma_0), C(\Gamma_0)) \cong H(\bigcirc_m)\{q^{m(N-m)}\} \langle m \rangle \cong C(\emptyset)\{ \begin{bmatrix} N \\ m \end{bmatrix} q^{m(N-m)} \}.$$

And the lemma follows. \square

Definition 7.31. Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 19. Associate to the loop addition a homogeneous morphism

$$\psi : C(\Gamma_0) \rightarrow C(\Gamma_1)$$

of quantum degree $-n(N-n) + mn$ not homotopic to 0.

Associate to the loop removal a homogeneous morphism

$$\bar{\psi} : C(\Gamma_1) \rightarrow C(\Gamma_0)$$

of quantum degree $-n(N-n) + mn$ not homotopic to 0.

By Lemma 7.30, ψ and $\bar{\psi}$ are well defined up to homotopy and scaling. Both of them have \mathbb{Z}_2 -grading n .

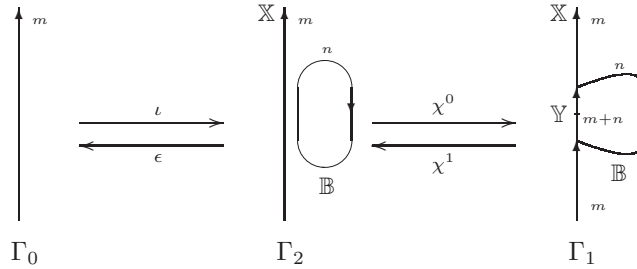


FIGURE 20.

The above definitions of ψ and $\bar{\psi}$ here are implicit. Next we give explicit constructions of ψ and $\bar{\psi}$. Consider the diagram in Figure 20, where χ^0, χ^1 are the morphisms given by proposition 7.20, ι, ϵ are the morphisms induced by the apparent circle creation and annihilation. Then $\chi^0 \circ \iota : C(\Gamma_0) \rightarrow C(\Gamma_1)$ and $\epsilon \circ \chi^1 : C(\Gamma_1) \rightarrow C(\Gamma_0)$ are both homogeneous morphisms of \mathbb{Z}_2 -degree n and quantum degree $-n(N-n) + mn$.

Proposition 7.32. $\psi \approx \chi^0 \circ \iota$, $\bar{\psi} \approx \epsilon \circ \chi^1$. Moreover, we have

$$(7.26) \quad \bar{\psi} \circ \mathfrak{m}(S_\mu(\mathbb{B})) \circ \psi \approx \begin{cases} \mathrm{id}_{C(\Gamma_0)} & \text{if } \mu = \lambda_{n, N-m-n}, \\ 0 & \text{if } |\mu| < n(N-m-n), \end{cases}$$

$$(7.27) \quad \bar{\psi} \circ \mathfrak{m}(S_\mu(\mathbb{Y})) \circ \psi \approx \begin{cases} \mathrm{id}_{C(\Gamma_0)} & \text{if } \mu = \lambda_{n, N-m-n}, \\ 0 & \text{if } |\mu| < n(N-m-n), \end{cases}$$

where $\mathfrak{m}(*)$ is the morphism given by the multiplication by $*$.

Proof. To prove $\psi \approx \chi^0 \circ \iota$ and $\bar{\psi} \approx \epsilon \circ \chi^1$, we only need to show that $\chi^0 \circ \iota$ and $\epsilon \circ \chi^1$ are not homotopic to 0. We prove this by showing that

$$(7.28) \quad \epsilon \circ \chi^1 \circ \mathbf{m}(S_\mu(\mathbb{B})) \circ \chi^0 \circ \iota \approx \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \mu = \lambda_{n, N-m-n}, \\ 0 & \text{if } |\mu| < n(N-m-n), \end{cases}$$

which also implies (7.26).

Note that the lowest non-vanishing quantum grading of $\text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_0))$ is 0 and, if $|\mu| < n(N-m-n)$, then the quantum degree of $\epsilon \circ \chi^1 \circ \mathbf{m}(S_\mu(\mathbb{B})) \circ \chi^0 \circ \iota$ is negative. This implies that $\epsilon \circ \chi^1 \circ \mathbf{m}(S_\mu(\mathbb{B})) \circ \chi^0 \circ \iota \simeq 0$ if $|\mu| < n(N-m-n)$. Now consider the case $\mu = \lambda_{n, N-m-n}$. By Proposition 7.20, We have

$$\begin{aligned} & \epsilon \circ \chi^1 \circ \mathbf{m}(S_{\lambda_{n, N-m-n}}(\mathbb{B})) \circ \chi^0 \circ \iota \\ &= \epsilon \circ \mathbf{m}(S_{\lambda_{n, N-m-n}}(\mathbb{B})) \circ \chi^1 \circ \chi^0 \circ \iota \\ &= \epsilon \circ \mathbf{m}(S_{\lambda_{n, N-m-n}}(\mathbb{B})) \cdot \sum_{\lambda \in \Lambda_{n, m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{X}) S_{\lambda^c}(\mathbb{B}) \circ \iota \\ &= \sum_{\lambda \in \Lambda_{n, m}} (-1)^{|\lambda|} S_{\lambda'}(\mathbb{X}) \cdot \epsilon \circ \mathbf{m}(S_{\lambda_{n, N-m-n}}(\mathbb{B})) \cdot S_{\lambda^c}(\mathbb{B}) \circ \iota \end{aligned}$$

where $\Lambda_{n, m} = \{\mu \mid \mu \leq \lambda_{n, m}\} = \{\mu = (\mu_1 \geq \dots \geq \mu_n) \mid l(\mu) \leq n, \mu_1 \leq m\}$, $\lambda' \in \Lambda_{m, n}$ is the conjugate of λ , and λ^c is the complement of λ in $\Lambda_{n, m}$, i.e., if $\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \Lambda_{n, m}$, then $\lambda^c = (m - \lambda_n \geq \dots \geq m - \lambda_1)$. By Corollary 7.8, we have, for $\lambda \in \Lambda_{n, m}$,

$$\epsilon \circ \mathbf{m}(S_{\lambda_{n, N-m-n}}(\mathbb{B})) \cdot S_{\lambda^c}(\mathbb{B}) \circ \iota \approx \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \lambda = (0 \geq \dots \geq 0), \\ 0 & \text{if } \lambda \neq (0 \geq \dots \geq 0). \end{cases}$$

This completes the proof for (7.28). Thus, we have proved $\psi \approx \chi^0 \circ \iota$, $\bar{\psi} \approx \epsilon \circ \chi^1$ and (7.26).

It remains to prove (7.27). Note that $\mathbf{m}(S_\mu(\mathbb{Y})) \simeq \mathbf{m}(S_\mu(\mathbb{B} \cup \mathbb{X}))$ as endomorphisms of $C(\Gamma_1)$ and $S_\mu(\mathbb{B} \cup \mathbb{X}) = S_\mu(\mathbb{B}) + F_\mu(\mathbb{B}, \mathbb{X})$, where $F_\mu(\mathbb{B}, \mathbb{X}) \in \text{Sym}(\mathbb{B}|\mathbb{X})$ and its total degree in \mathbb{B} is strictly less than $2|\mu|$. Then, by (7.26) for any partition μ with $|\mu| \leq n(N-m-n)$, we have

$$\bar{\psi} \circ \mathbf{m}(S_\mu(\mathbb{Y})) \circ \psi \simeq \bar{\psi} \circ \mathbf{m}(S_\mu(\mathbb{B} \cup \mathbb{X})) \circ \psi \simeq \bar{\psi} \circ \mathbf{m}(S_\mu(\mathbb{B}) + F_\mu(\mathbb{B}, \mathbb{X})) \circ \psi \simeq \bar{\psi} \circ \mathbf{m}(S_\mu(\mathbb{B})) \circ \psi.$$

So (7.27) follows from (7.26). \square

7.8. Saddle move. Next we define the morphism η induced by the saddle move in Figure 21. Unlike the morphisms in the previous subsections, we will not give an explicit form of η . Instead, we prove two composition formulas of η , which are all we need to know about η in this paper.

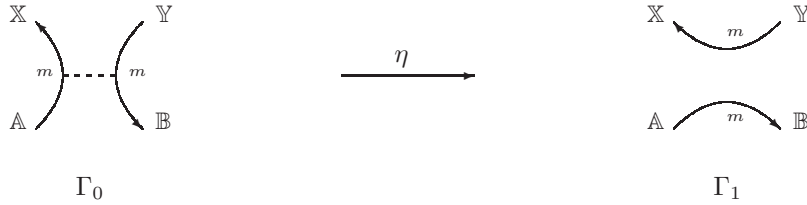


FIGURE 21.

Lemma 7.33. *Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 21. Then, as bigraded vector spaces over \mathbb{C} ,*

$$\mathrm{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1)) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} q^{2m(N-m)} \right\} \langle m \rangle.$$

In particular, the subspace of $\mathrm{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1))$ of homogeneous elements of quantum degree $m(N-m)$ is 1-dimensional.

Proof. Let \bigcirc_m be a circle colored by m with 4 marked points. By lemmas 2.11, 2.12 and 2.13, one can see that $\mathrm{Hom}(C(\Gamma_0), C(\Gamma_1)) \cong C(\bigcirc_m) \{q^{2m(N-m)}\}$. The lemma follows from this and Corollary 6.1. \square

Definition 7.34. Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 21. Associate to the saddle move $\Gamma_0 \rightsquigarrow \Gamma_1$ a homogeneous morphism

$$\eta : C(\Gamma_0) \rightarrow C(\Gamma_1)$$

of quantum degree $m(N-m)$ not homotopic to 0. By Lemma 7.33, η is well defined up to homotopy and scaling, and $\deg_{\mathbb{Z}_2} \eta = m$.

7.9. The first composition formula. In this subsection, we prove that the change in Figure 22 induces, up to homotopy and scaling, the identity map of the matrix factorization. Topologically, this means that a pair of canceling 0- and 1-handles induce the identity morphism.

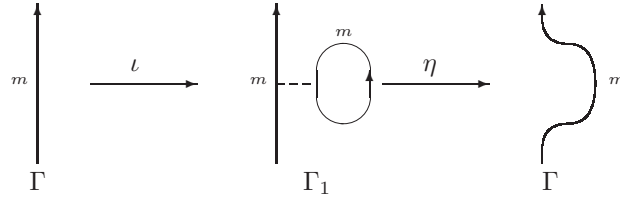


FIGURE 22.

Lemma 7.35. *Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 21. Then under the identification*

$$\mathrm{Hom}(C(\Gamma_0), C(\Gamma_1)) \cong C(\Gamma_1) \otimes_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})} C(\Gamma_0)_{\bullet} \cong \begin{pmatrix} * & X_1 - Y_1 \\ \dots & \dots \\ * & X_m - Y_m \\ * & B_1 - A_1 \\ \dots & \dots \\ * & B_m - A_m \\ A_1 - X_1 & * \\ \dots & \dots \\ A_m - X_m & * \\ Y_1 - B_1 & * \\ \dots & \dots \\ Y_m - B_m & * \end{pmatrix}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})},$$

where X_j is the j -th elementary symmetric polynomial in \mathbb{X} and so on, we have

$$\eta \approx \rho + \left(\sum_{\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + \sum_{j=1}^{m-1} (m-j)\varepsilon_j} 1_{\varepsilon} \otimes 1_{\bar{\varepsilon}} \right) \otimes 1_{\underbrace{(1, 1, \dots, 1)}_{2m}},$$

where ρ is of the form

$$\rho = \sum_{\varepsilon_1, \varepsilon_2 \in I^m, \varepsilon_3 \in I^{2m}, \varepsilon_3 \neq (1, 1, \dots, 1)} f_{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{\varepsilon_3}.$$

Proof. Write $R_0 = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$, and

$$R_k = \begin{cases} R/(A_1 - X_1, \dots, A_k - X_k) & \text{if } 1 \leq k \leq m, \\ R/(A_1 - X_1, \dots, A_m - X_m, Y_1 - B_1, \dots, Y_{k-m} - B_{k-m}) & \text{if } m+1 \leq k \leq 2m. \end{cases}$$

Define

$$M_k = \begin{pmatrix} * & X_1 - Y_1 \\ \dots & \dots \\ * & X_m - Y_m \\ * & B_1 - A_1 \\ \dots & \dots \\ * & B_m - A_m \\ A_{k+1} - X_{k+1} & * \\ \dots & \dots \\ A_m - X_m & * \\ Y_1 - B_1 & * \\ \dots & \dots \\ Y_m - B_m & * \end{pmatrix}_{R_k} \quad \text{if } 0 \leq k \leq m-1,$$

$$M_k = \begin{pmatrix} * & X_1 - Y_1 \\ \dots & \dots \\ * & X_m - Y_m \\ * & B_1 - A_1 \\ \dots & \dots \\ * & B_m - A_m \\ Y_{k-m+1} - B_{k-m+1} & * \\ \dots & \dots \\ Y_m - B_m & * \end{pmatrix}_{R_k} \quad \text{if } m \leq k \leq 2m-1.$$

and

$$M_{2m} = \begin{pmatrix} * & X_1 - Y_1 \\ \dots & \dots \\ * & X_m - Y_m \\ * & B_1 - A_1 \\ \dots & \dots \\ * & B_m - A_m \end{pmatrix}_{R_{2m}} \cong C(\Gamma),$$

where Γ is a circle colored by m with two marked points shown in Figure 11. Then $\text{Hom}_{HFM}(C(\Gamma_0), C(\Gamma_1))$ can be computed by the following homotopy

$$\begin{aligned} \text{Hom}(C(\Gamma_0), C(\Gamma_1)) &\cong M_0 \simeq \dots \simeq M_k \{q^{n_k}\} \langle k \rangle \simeq \dots \simeq M_{2m} \{q^{n_{2m}}\} \langle 2m \rangle \\ &\cong C(\Gamma) \{q^{2m(N-m)}\} \simeq C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} q^{2m(N-m)} \right\} \langle m \rangle, \end{aligned}$$

where n_k can be inductively computed using Corollary 2.25. In particular, $n_{2m} = 2m(N-m)$. Let $\eta_k \in M_k$ be the image of η under the above homotopy. Then η_k represents, up to scaling, the unique homology class in $H(M_k)$ of quantum degree $m(N-m) - q^{n_k}$.

By Lemma 6.5,

$$\eta_{2m} \approx \sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \in M_{2m},$$

where $s(\varepsilon) = \sum_{j=1}^{m-1} (m-j)\varepsilon_j$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in I^m$. Assume that

$$\eta_k \approx \rho_k + \left(\sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \right) \otimes 1_{\underbrace{(1, 1, \dots, 1)}_{2m-k}} \in M_k,$$

where ρ_k is of the form

$$\rho_k = \sum_{\varepsilon_1, \varepsilon_2 \in I^m, \varepsilon_3 \in I^{2m-k}, \varepsilon_3 \neq (1, 1, \dots, 1)} f_{k, (\varepsilon_1, \varepsilon_2, \varepsilon_3)} 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{\varepsilon_3}.$$

Note that

$$\tilde{\eta}_k \approx \tilde{\rho}_k + \left(\sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \right) \otimes 1_{\underbrace{(1, 1, \dots, 1)}_{2m-k+1}}$$

is a chain in M_{k-1} mapped to η_k under the homotopy

$$M_{k-1}\{q^{n_{k-1}}\} \langle k-1 \rangle \xrightarrow{\cong} M_k\{q^{n_k}\} \langle k \rangle,$$

where

$$\tilde{\rho}_k = \sum_{\varepsilon_1, \varepsilon_2 \in I^m, \varepsilon_3 \in I^{2m-k}, \varepsilon_3 \neq (1, 1, \dots, 1)} f_{k, (\varepsilon_1, \varepsilon_2, \varepsilon_3)} 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{(1, \varepsilon_3)}.$$

Then, by Corollary 2.25 and Remark 2.21, we have that

$$\begin{aligned} \eta_{k-1} &\approx \tilde{\eta}_k - h \circ d(\tilde{\eta}_k) \\ &= \tilde{\rho}_k - h \circ d(\tilde{\eta}_k) + \left(\sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \right) \otimes 1_{\underbrace{(1, 1, \dots, 1)}_{2m-k+1}}. \end{aligned}$$

See the proof of Proposition 2.20 for the definition of h and note the slightly different setup here. (We are eliminating a row here by mod out its first entry rather than the second.) By the definition of h , (again, note the difference in the setup,) it is easy to see that $h \circ d(\tilde{\eta}_k)$ is of the form

$$h \circ d(\tilde{\eta}_k) = \sum_{\varepsilon_1, \varepsilon_2 \in I^m, \varepsilon_3 \in I^{2m-k}} g_{k, (\varepsilon_1, \varepsilon_2, \varepsilon_3)} 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{(0, \varepsilon_3)}.$$

Therefore, $\rho_{k-1} := \tilde{\rho}_k - h \circ d(\tilde{\eta}_k)$ is of the form

$$\rho_{k-1} = \sum_{\varepsilon_1, \varepsilon_2 \in I^m, \varepsilon_3 \in I^{2m-k+1}, \varepsilon_3 \neq (1, 1, \dots, 1)} f_{k-1, (\varepsilon_1, \varepsilon_2, \varepsilon_3)} 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{\varepsilon_3}.$$

Thus, we have inductively constructed a $\rho = \rho_0 \in M_0$ of the form

$$\rho = \sum_{\varepsilon_1, \varepsilon_2 \in I^m, \varepsilon_3 \in I^{2m}, \varepsilon_3 \neq (1, 1, \dots, 1)} f_{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{\varepsilon_3}$$

such that

$$\eta \approx \rho + \left(\sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}} \right) \otimes 1_{\underbrace{(1, 1, \dots, 1)}_{2m}}.$$

□

Proposition 7.36. *Let Γ and Γ_1 be the colored MOY graphs in Figure 22, $\iota : C(\Gamma) \rightarrow C(\Gamma_1)$ the morphism associated to the circle creation and $\eta : C(\Gamma_1) \rightarrow C(\Gamma)$ the morphism associated to the saddle move. Then $\eta \circ \iota \approx \text{id}_{C(\Gamma)}$.*

Proof. From the proof of Lemma 6.5, we know that

$$\text{Hom}_{HMF}(C(\Gamma), C(\Gamma)) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} q^{m(N-m)} \right\}.$$

In particular, the subspace of $\text{Hom}_{HMF}(C(\Gamma), C(\Gamma))$ of elements of quantum degree 0 is 1-dimensional and spanned by $\text{id}_{C(\Gamma)}$. Note that the quantum degree of $\eta \circ \iota$ is 0. So, to prove that $\eta \circ \iota \approx \text{id}_{C(\Gamma)}$, we only need to show that $\eta \circ \iota$ is not homotopic to 0. We do so by identifying the two ends of Γ and showing that $\eta_* \circ \iota_* \neq 0$.

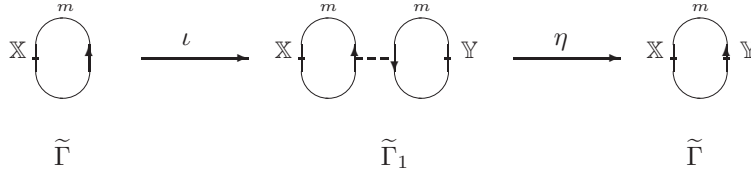


FIGURE 23.

Identify the two end points in each of the colored MOY graphs in Figure 22 and put markings on them as in Figure 23. Denote by $\tilde{\Gamma}$ and $\tilde{\Gamma}_1$ the resulting colored MOY graphs. Denote by \mathfrak{G} the generating class of $H(\tilde{\Gamma})$ and $\mathfrak{G}_X, \mathfrak{G}_Y$ the generating classes of the homology the two circles in $H(\tilde{\Gamma}_1)$. Then $\iota_*(\mathfrak{G}) \propto \mathfrak{G}_X \otimes \mathfrak{G}_Y$. And, by lemmas 2.11, 6.4, 6.5 and 7.35, $\eta_*(\mathfrak{G}_X \otimes \mathfrak{G}_Y) \propto \mathfrak{G}$. Thus, $\eta_* \circ \iota_*(\mathfrak{G}) \propto \mathfrak{G}$. This shows that $\eta \circ \iota$ is not homotopic to 0 and, therefore, $\eta \circ \iota \approx \text{id}_{C(\Gamma)}$. \square

Remark 7.37. From the proof of Proposition 7.36, we can see that η gives $H(\tilde{\Gamma})$ a ring structure and $H(\tilde{\Gamma})\{q^{m(N-m)}\} \cong H^*(G_{m,N}; \mathbb{C})$ as \mathbb{Z} -graded \mathbb{C} -algebras.

7.10. The second composition formula. In this subsection, we show that the change in Figure 24 also induces, up to homotopy and scaling, the identity map. Topologically, this means that a pair of canceling 1- and 2-handles induce the identity morphism. The key to the proof is a good choice of the entries in the left column of the matrix factorizations involved. Our choice is given in the following lemma.

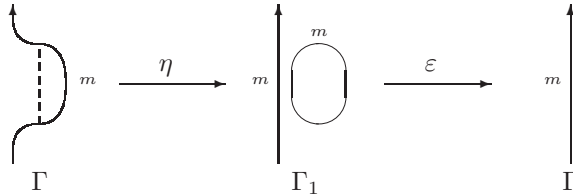


FIGURE 24.

Lemma 7.38. *Let \mathbb{X}, \mathbb{Y} be disjoint alphabets, each having m ($\leq N$) indeterminants. For $j = 1, \dots, m$, define*

$$\begin{aligned} U_j(\mathbb{X}, \mathbb{Y}) &= (-1)^{j-1} p_{N+1-j}(\mathbb{Y}) + \sum_{k=1}^m (-1)^{k+j} j X_k h_{N+1-k-j}(\mathbb{Y}) \\ &\quad + \sum_{k=1}^m \sum_{l=1}^m (-1)^{k+l} l X_k X_l \xi_{N+1-k-l,j}(\mathbb{X}, \mathbb{Y}), \end{aligned}$$

where X_j and Y_j are the j -th elementary symmetric polynomials in \mathbb{X} and \mathbb{Y} , and

$$\xi_{n,j}(\mathbb{X}, \mathbb{Y}) = \frac{h_{m,n}(Y_1, \dots, Y_{j-1}, X_j, \dots, X_m) - h_{m,n}(Y_1, \dots, Y_j, X_{j+1}, \dots, X_m)}{X_j - Y_j}.$$

Then $U_j(\mathbb{X}, \mathbb{Y})$ is homogeneous of degree $2(N+1-j)$ and

$$\sum_{j=1}^m (X_j - Y_j) U_j(\mathbb{X}, \mathbb{Y}) = p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y}).$$

Proof. The claims about the homogeneity and degree of $U_j(\mathbb{X}, \mathbb{Y})$ are obviously true. So we only prove the last equation. Since $N \geq m$, by Newton's Identity (4.3), we have that

$$\begin{aligned} &p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y}) \\ &= \sum_{k=1}^m (-1)^{k-1} (X_k p_{N+1-k}(\mathbb{X}) - Y_k p_{N+1-k}(\mathbb{Y})) \\ &= \sum_{k=1}^m (-1)^{k-1} (X_k - Y_k) p_{N+1-k}(\mathbb{Y}) + \sum_{k=1}^m (-1)^{k-1} X_k (p_{N+1-k}(\mathbb{X}) - p_{N+1-k}(\mathbb{Y})). \end{aligned}$$

By (4.2),

$$\begin{aligned} &p_{N+1-k}(\mathbb{X}) - p_{N+1-k}(\mathbb{Y}) \\ &= \sum_{l=1}^m (-1)^{l-1} l (X_l h_{N+1-k-l}(\mathbb{X}) - Y_l h_{N+1-k-l}(\mathbb{Y})) \\ &= \sum_{l=1}^m (-1)^{l-1} l (X_l - Y_l) h_{N+1-k-l}(\mathbb{Y}) + \sum_{l=1}^m (-1)^{l-1} l X_l (h_{N+1-k-l}(\mathbb{X}) - h_{N+1-k-l}(\mathbb{Y})) \\ &= \sum_{l=1}^m (-1)^{l-1} l (X_l - Y_l) h_{N+1-k-l}(\mathbb{Y}) + \sum_{l=1}^m (-1)^{l-1} l X_l \sum_{j=1}^m \xi_{N+1-k-l,j}(\mathbb{X}, \mathbb{Y}) (X_j - Y_j). \end{aligned}$$

Substituting this back into the first equation, we get

$$\begin{aligned} &p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y}) \\ &= \sum_{k=1}^m (-1)^{k-1} (X_k - Y_k) p_{N+1-k}(\mathbb{Y}) + \sum_{k=1}^m (-1)^{k-1} X_k \sum_{l=1}^m (-1)^{l-1} l (X_l - Y_l) h_{N+1-k-l}(\mathbb{Y}) \\ &\quad + \sum_{k=1}^m (-1)^{k-1} X_k \sum_{l=1}^m (-1)^{l-1} l X_l \sum_{j=1}^m \xi_{N+1-k-l,j}(\mathbb{X}, \mathbb{Y}) (X_j - Y_j) \\ &= \sum_{j=1}^m (X_j - Y_j) U_j(\mathbb{X}, \mathbb{Y}). \end{aligned}$$

□

The next lemma is a special case of Remark 2.21.

Lemma 7.39. *Let R be a graded commutative unital \mathbb{C} -algebra, and X an homogeneous indeterminant over R . Assume that $f_{1,0}(X), f_{1,1}(X), \dots, f_{k,0}(X), f_{k,1}(X)$ are homogeneous elements in $R[X]$ such that*

$$\begin{aligned} \deg f_{j,0}(X) + \deg f_{j,1}(X) &= 2N + 2, \\ \sum_{j=1}^k f_{j,0}(X) f_{j,1}(X) &= 0. \end{aligned}$$

Suppose that $f_{1,1}(X) = X - A$, where $A \in R$ is a homogeneous element of degree $\deg A = \deg X$. Define

$$M = \begin{pmatrix} f_{1,0}(X) & f_{1,1}(X) \\ f_{2,0}(X) & f_{2,1}(X) \\ \dots & \dots \\ f_{k,0}(X) & f_{k,1}(X) \end{pmatrix}_{R[X]} \quad \text{and} \quad M' = \begin{pmatrix} f_{2,0}(A) & f_{2,1}(A) \\ f_{3,0}(A) & f_{3,1}(A) \\ \dots & \dots \\ f_{k,0}(A) & f_{k,1}(A) \end{pmatrix}_R.$$

Then M and M' are homotopic graded chain complexes. Let $F : M \rightarrow M'$ be the quasi-isomorphism from the proof of Proposition 2.20. If

$$\alpha = \sum_{\varepsilon \in I^{k-1}} a_\varepsilon 1_\varepsilon$$

is a cycle in M' , where $a_\varepsilon \in R$, then

$$\tilde{\alpha} = \sum_{\varepsilon \in I^{k-1}} a_\varepsilon 1_{(0,\varepsilon)} - \sum_{\varepsilon=(\varepsilon_2,\dots,\varepsilon_k) \in I^{k-1}} a_\varepsilon \left(\sum_{j=2}^k (-1)^{|(0,\varepsilon)|_j} g_{j,\varepsilon_j}(X) 1_{(1,\varepsilon_2,\dots,\varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)} \right),$$

where $| (0, \varepsilon) |_j = \sum_{l=2}^{j-1} \varepsilon_l$ and $g_{j,\varepsilon_j}(X) = \frac{f_{j,\varepsilon_j}(X) - f_{j,\varepsilon_j}(A)}{X - A}$, is a cycle in M and $F(\tilde{\alpha}) = \alpha$.

Proof. Let $\beta = \sum_{\varepsilon \in I^{k-1}} a_\varepsilon 1_{(0,\varepsilon)} \in M$. Then $F(\beta) = \alpha$. By Remark 2.21, we know that $d(\beta) \in \ker F$, $\beta - h \circ d(\beta)$ is a cycle in M and $F(\beta - h \circ d(\beta)) = \alpha$, where $h : \ker F \rightarrow \ker F$ is defined in the proof of Proposition 2.20. But

$$\begin{aligned} h \circ d(\beta) &= h \circ d \left(\sum_{\varepsilon \in I^{k-1}} a_\varepsilon 1_{(0,\varepsilon)} \right) \\ &= h \left(\sum_{\varepsilon=(\varepsilon_2,\dots,\varepsilon_k) \in I^{k-1}} a_\varepsilon (f_{1,0}(X) 1_{(1,\varepsilon)} + \sum_{j=2}^k (-1)^{|(0,\varepsilon)|_j} f_{j,\varepsilon_j}(X) 1_{(0,\varepsilon_2,\dots,\varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)}) \right). \end{aligned}$$

By the definition of h , we know that $h(1_{(1,\varepsilon)}) = 0$. Moreover, since α is a cycle in M' , we have

$$0 = d\alpha = \sum_{\varepsilon=(\varepsilon_2,\dots,\varepsilon_k) \in I^{k-1}} a_\varepsilon \left(\sum_{j=2}^k (-1)^{|(0,\varepsilon)|_j} f_{j,\varepsilon_j}(A) 1_{(\varepsilon_2,\dots,\varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)} \right).$$

So, in M , we have

$$0 = \sum_{\varepsilon=(\varepsilon_2,\dots,\varepsilon_k) \in I^{k-1}} a_\varepsilon \left(\sum_{j=2}^k (-1)^{|(0,\varepsilon)|_j} f_{j,\varepsilon_j}(A) 1_{(0,\varepsilon_2,\dots,\varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)} \right).$$

Thus,

$$\begin{aligned} h \circ d(\beta) &= h\left(\sum_{\varepsilon=(\varepsilon_2, \dots, \varepsilon_k) \in I^{k-1}} a_\varepsilon \left(\sum_{j=2}^k (-1)^{|(0, \varepsilon)|_j} (f_{j, \varepsilon_j}(X) - f_{j, \varepsilon_j}(A)) 1_{(0, \varepsilon_2, \dots, \varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)}\right)\right) \\ &= \sum_{\varepsilon=(\varepsilon_2, \dots, \varepsilon_k) \in I^{k-1}} a_\varepsilon \left(\sum_{j=2}^k (-1)^{|(0, \varepsilon)|_j} g_{j, \varepsilon_j}(X) 1_{(1, \varepsilon_2, \dots, \varepsilon_{j-1}, \overline{\varepsilon_j}, \varepsilon_{j+1}, \dots, \varepsilon_k)}\right), \end{aligned}$$

where the last equation comes from the definition of h . This shows that $\beta - h \circ d(\beta) = \tilde{\alpha}$ and proves the lemma. \square

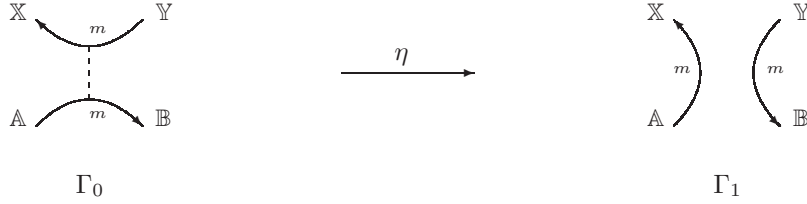


FIGURE 25.

Let Γ_0 and Γ_1 be the colored MOY graphs in Figure 25. And $\eta : C(\Gamma_0) \rightarrow C(\Gamma_1)$ the morphism induced by the saddle move. We have

$$\text{Hom}(C(\Gamma_0), C(\Gamma_1)) \cong C(\Gamma) \otimes_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})} C(\Gamma_0)_\bullet \cong \begin{pmatrix} V_1(\mathbb{A}, \mathbb{X}) & X_1 - A_1 \\ \dots & \dots \\ V_m(\mathbb{A}, \mathbb{X}) & X_m - A_m \\ V_1(\mathbb{B}, \mathbb{Y}) & B_1 - Y_1 \\ \dots & \dots \\ V_m(\mathbb{B}, \mathbb{Y}) & B_m - Y_m \\ A_1 - B_1 & U_1(\mathbb{A}, \mathbb{B}) \\ \dots & \dots \\ A_m - B_m & U_m(\mathbb{A}, \mathbb{B}) \\ Y_1 - X_1 & U_1(\mathbb{X}, \mathbb{Y}) \\ \dots & \dots \\ Y_m - X_m & U_m(\mathbb{X}, \mathbb{Y}) \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})},$$

where X_j is the j -th elementary symmetric polynomial in \mathbb{X} , U_j is given by Lemma 7.38, and

$$V_j(\mathbb{X}, \mathbb{Y}) := \frac{p_{m, N+1}(Y_1, \dots, Y_{j-1}, X_j, \dots, X_m) - p_{m, N+1}(Y_1, \dots, Y_j, X_{j+1}, \dots, X_m)}{X_j - Y_j}.$$

By definition, it is easy to see that

$$(7.29) \quad \frac{\partial}{\partial X_k} V_j(\mathbb{X}, \mathbb{Y}) = 0 \text{ if } j > k,$$

$$(7.30) \quad \frac{\partial}{\partial Y_k} V_j(\mathbb{X}, \mathbb{Y}) = 0 \text{ if } j < k.$$

Set $R_0 = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B}) = \mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_m, A_1, \dots, A_m, B_1, \dots, B_m]$,
and, for $1 \leq k \leq m$,

$$\begin{aligned} R_k &= R_0/(X_1 - A_1, \dots, X_k - A_k) \\ &\cong \mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_m, A_{k+1}, \dots, A_m, B_1, \dots, B_m], \\ R_{m+k} &= R_0/(X_1 - A_1, \dots, X_m - A_m, B_1 - Y_1, \dots, B_k - Y_k) \\ &\cong \mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_m, B_{k+1}, \dots, B_m]. \end{aligned}$$

Define

$$\begin{aligned} M_k &= \begin{pmatrix} V_{k+1}(\mathbb{A}, \mathbb{X}) & X_{k+1} - A_{k+1} \\ \dots & \dots \\ V_m(\mathbb{A}, \mathbb{X}) & X_m - A_m \\ V_1(\mathbb{B}, \mathbb{Y}) & B_1 - Y_1 \\ \dots & \dots \\ V_m(\mathbb{B}, \mathbb{Y}) & B_m - Y_m \\ A_1 - B_1 & U_1(\mathbb{A}, \mathbb{B}) \\ \dots & \dots \\ A_m - B_m & U_m(\mathbb{A}, \mathbb{B}) \\ Y_1 - X_1 & U_1(\mathbb{X}, \mathbb{Y}) \\ \dots & \dots \\ Y_m - X_m & U_m(\mathbb{X}, \mathbb{Y}) \end{pmatrix}_{R_k} \quad \text{for } k = 0, 1, \dots, m-1, \\ M_{m+k} &= \begin{pmatrix} V_{k+1}(\mathbb{B}, \mathbb{Y}) & B_{k+1} - Y_{k+1} \\ \dots & \dots \\ V_m(\mathbb{B}, \mathbb{Y}) & B_m - Y_m \\ A_1 - B_1 & U_1(\mathbb{A}, \mathbb{B}) \\ \dots & \dots \\ A_m - B_m & U_m(\mathbb{A}, \mathbb{B}) \\ Y_1 - X_1 & U_1(\mathbb{X}, \mathbb{Y}) \\ \dots & \dots \\ Y_m - X_m & U_m(\mathbb{X}, \mathbb{Y}) \end{pmatrix}_{R_{m+k}} \quad \text{for } k = 0, 1, \dots, m-1, \\ M_{2m} &= \begin{pmatrix} X_1 - Y_1 & U_1(\mathbb{X}, \mathbb{Y}) \\ \dots & \dots \\ X_m - Y_m & U_m(\mathbb{X}, \mathbb{Y}) \\ Y_1 - X_1 & U_1(\mathbb{X}, \mathbb{Y}) \\ \dots & \dots \\ Y_m - X_m & U_m(\mathbb{X}, \mathbb{Y}) \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y})}. \end{aligned}$$

By Proposition 2.19, $M_0 \simeq M_1 \simeq \dots \simeq M_{2m}$. Let η_k be the image of η in M_k . Then, use method in the proof of Lemma 6.5, one can check that

$$\eta_{2m} \approx \sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + |\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}},$$

where $s(\varepsilon) := \sum_{j=1}^{m-1} (m-j)\varepsilon_j$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in I^m$.

Next, we apply Lemma 7.39 to find a cycle representing η in M_0 .

Write

$$\begin{aligned} \theta_{j,0}(X_1, \dots, X_m, Y_1, \dots, Y_m) &= X_j - Y_j, \\ \theta_{j,1}(X_1, \dots, X_m, Y_1, \dots, Y_m) &= U_j(\mathbb{X}, \mathbb{Y}). \end{aligned}$$

And define, for $k = 1, \dots, m$, $\varepsilon \in \mathbb{Z}_2$,

$$\begin{aligned}\Theta_{j,\varepsilon}^k &= \frac{\theta_{j,\varepsilon}(X_1, \dots, X_{k-1}, A_k, \dots, A_m, B_1, \dots, B_m) - \theta_{j,\varepsilon}(X_1, \dots, X_k, A_{k+1}, \dots, A_m, B_1, \dots, B_m)}{X_k - A_k}, \\ \Theta_{j,\varepsilon}^{m+k} &= \frac{\theta_{j,\varepsilon}(X_1, \dots, X_m, Y_1, \dots, Y_{k-1}, B_k, \dots, B_m) - \theta_{j,\varepsilon}(X_1, \dots, X_m, Y_1, \dots, Y_k, B_{k+1}, \dots, B_m)}{B_k - Y_k}.\end{aligned}$$

It is easy to see that, for $1 \leq k, j \leq m$,

$$(7.31) \quad \Theta_{j,0}^k = \Theta_{j,0}^{m+k} = \begin{cases} -1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

In the following computation, we shall call an element of M_0 an irrelevant term if it is of the form $c \cdot 1_{\varepsilon_1} \otimes 1_{\varepsilon_2} \otimes 1_{\varepsilon_3}$ where $c \in R_0$, $\varepsilon_1 \in I^{2m}$ and $\varepsilon_2, \varepsilon_3 \in I^m$ such that either $\varepsilon_1 \neq (1, 1, \dots, 1)$ or $\varepsilon_2 \neq \overline{\varepsilon_3}$.

Define \mathcal{F} to be the set of functions from $\{1, 2, \dots, 2m\}$ to $\{1, 2, \dots, m\}$ and

$$\begin{aligned}\mathcal{F}_{\text{even}} &= \{f \in \mathcal{F} \mid \#f^{-1}(j) \text{ is even for } j = 1, 2, \dots, m\}, \\ \mathcal{F}_2 &= \{f \in \mathcal{F} \mid \#f^{-1}(j) = 2 \text{ for } j = 1, 2, \dots, m\}.\end{aligned}$$

For $f \in \mathcal{F}$, $k = 1, 2, \dots, 2m$, define

$$\begin{aligned}\nu_{f,k} &= \#\{k' \mid k < k' \leq 2m, f(k') < f(k)\}, \\ \nu_f &= \sum_{k=1}^{2m} \nu_{f,k}, \\ \mu_{f,k} &= \#\{k' \mid k < k' \leq 2m, f(k') = f(k)\}.\end{aligned}$$

For $f \in \mathcal{F}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in I^m$, define $\varphi_f(\varepsilon) = (e_1, \dots, e_m) \in I^m$, where $e_j \in I$ satisfies

$$e_j \equiv \varepsilon_j + \#\{k \mid 1 \leq k \leq 2m, f(k) = j\} \pmod{2}.$$

Applying Lemma 7.39 repeatedly, we get that

$$\begin{aligned}\eta_0 &\approx \sum_{\varepsilon \in I^m} (-1)^{\frac{|\overline{\varepsilon}|(|\overline{\varepsilon}|-1)}{2} + |\overline{\varepsilon}| + s(\overline{\varepsilon}) + 2m} \sum_{f \in \mathcal{F}} \left(\prod_{k=1}^{2m} (-1)^{|\varepsilon|_{f(k)} + \nu_{f,k}} \Theta_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right) 1_{(1, \dots, 1)} \otimes 1_{\varphi_f(\varepsilon)} \otimes 1_{\overline{\varepsilon}} \\ &\quad + \text{irrelevant terms},\end{aligned}$$

where ε_j is the j -th entry in ε . Note that, if $f \notin \mathcal{F}_{\text{even}}$, then $\varphi_f(\varepsilon) \neq \varepsilon$ and the corresponding term in the above sum is also irrelevant. So we can simplify the above and get

$$\begin{aligned}\eta_0 &\approx \sum_{\varepsilon \in I^m} (-1)^{\frac{|\overline{\varepsilon}|(|\overline{\varepsilon}|-1)}{2} + |\overline{\varepsilon}| + s(\overline{\varepsilon})} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{\nu_f} \left(\prod_{k=1}^{2m} \Theta_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right) 1_{(1, \dots, 1)} \otimes 1_{\varepsilon} \otimes 1_{\overline{\varepsilon}} \\ &\quad + \text{irrelevant terms}.\end{aligned}$$

In Figure 25, identify the two end points of Γ_0 marked by \mathbb{X} and \mathbb{A} , and identify the two end points of Γ_0 marked by \mathbb{Y} and \mathbb{B} . This changes Γ_0 into $\tilde{\Gamma}$ in Figure 26. Similarly, by identifying the two end points of Γ_1 in Figure 25 marked by \mathbb{X} and \mathbb{A} and identifying the two end points of Γ_1 marked by \mathbb{Y} and \mathbb{B} , we change Γ_1 into $\tilde{\Gamma}_1$ in Figure 26. Let \mathfrak{G} be the generating class of $H(\tilde{\Gamma})$, and $\mathfrak{G}_{\mathbb{X}}$ and $\mathfrak{G}_{\mathbb{Y}}$ the

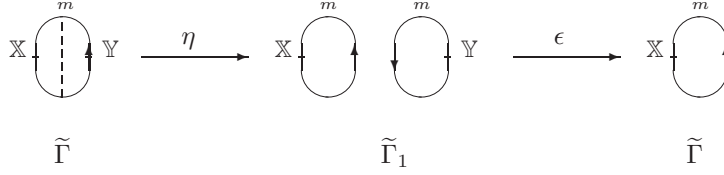


FIGURE 26.

generating classes of the homology of the two circles in $\tilde{\Gamma}_1$. By Lemma 6.5, \mathfrak{G} is represented in

$$C(\tilde{\Gamma}) = \begin{pmatrix} U_1(\mathbb{X}, \mathbb{Y}) & Y_1 - X_1 \\ \vdots & \vdots \\ U_m(\mathbb{X}, \mathbb{Y}) & Y_m - X_m \\ U_1(\mathbb{X}, \mathbb{Y}) & X_1 - Y_1 \\ \vdots & \vdots \\ U_m(\mathbb{X}, \mathbb{Y}) & X_m - Y_m \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y})}$$

by the cycle

$$G = \sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon)} 1_\varepsilon \otimes 1_{\bar{\varepsilon}}.$$

Define $\tilde{\Theta}_{j,0}^k$, $\tilde{\Theta}_{j,0}^{m+k}$, $\tilde{\Theta}_{j,1}^k$, $\tilde{\Theta}_{j,1}^{m+k}$ by substituting $A_1 = X_1, \dots, A_m = X_m$, $B_1 = Y_1, \dots, B_m = Y_m$ into $\Theta_{j,0}^k$, $\Theta_{j,0}^{m+k}$, $\Theta_{j,1}^k$, $\Theta_{j,1}^{m+k}$. Then, for $1 \leq k, j \leq m$,

$$(7.32) \quad \tilde{\Theta}_{j,0}^k = \tilde{\Theta}_{j,0}^{m+k} = \begin{cases} -1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

$$(7.33) \quad \tilde{\Theta}_{j,1}^k := \Theta_{j,1}^k|_{A_1=X_1, \dots, A_m=X_m, B_1=Y_1, \dots, B_m=Y_m} = -\frac{\partial}{\partial X_k} U_j(\mathbb{X}, \mathbb{Y}),$$

$$(7.34) \quad \tilde{\Theta}_{j,1}^{m+k} := \Theta_{j,1}^{m+k}|_{A_1=X_1, \dots, A_m=X_m, B_1=Y_1, \dots, B_m=Y_m} = \frac{\partial}{\partial Y_k} U_j(\mathbb{X}, \mathbb{Y}).$$

Using the formula for η_0 and lemmas 2.11, 2.12, we get that $\eta(G)$ is represented in

$$C(\tilde{\Gamma}_1) = \begin{pmatrix} V_1(\mathbb{X}, \mathbb{X}) & 0 \\ \vdots & \vdots \\ V_m(\mathbb{X}, \mathbb{X}) & 0 \\ V_1(\mathbb{Y}, \mathbb{Y}) & 0 \\ \vdots & \vdots \\ V_m(\mathbb{Y}, \mathbb{Y}) & 0 \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y})}$$

by the cycle

$$\begin{aligned} & \eta(G) \\ \approx & \sum_{\varepsilon \in I^m} (-1)^{\frac{|\varepsilon|(|\varepsilon|-1)}{2} + |\bar{\varepsilon}| + s(\bar{\varepsilon}) + \frac{|\varepsilon|(|\varepsilon|-1)}{2} + (m+1)|\varepsilon| + s(\varepsilon) + \frac{m(m-1)}{2}} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{\nu_f} \left(\prod_{k=1}^{2m} \tilde{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right) 1_{(1, \dots, 1)} \\ & + \text{irrelevant terms,} \end{aligned}$$

where the “irrelevant terms” are terms not of the form $c \cdot 1_{(1,\dots,1)}$. By definition, it is easy to see that

$$\begin{aligned} |\varepsilon| + |\bar{\varepsilon}| &= m, \\ s(\varepsilon) + s(\bar{\varepsilon}) &= \sum_{j=1}^{m-1} (m-j) = \frac{m(m-1)}{2}. \end{aligned}$$

Then one can check that

$$\begin{aligned} & \frac{|\bar{\varepsilon}|(|\bar{\varepsilon}| - 1)}{2} + |\bar{\varepsilon}| + s(\bar{\varepsilon}) + \frac{|\varepsilon|(|\varepsilon| - 1)}{2} + (m+1)|\varepsilon| + s(\varepsilon) + \frac{m(m-1)}{2} \\ \equiv & |\varepsilon|^2 + \frac{m(m+1)}{2} \equiv |\varepsilon| + \frac{m(m+1)}{2} \pmod{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta(G) &\approx (-1)^{\frac{m(m+1)}{2}} \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \tilde{\Theta}_{f^{(k)}, \varepsilon_{f^{(k)}} + \mu_{f,k}}^k \right) 1_{(1,\dots,1)} \\ &\quad + \text{irrelevant terms.} \end{aligned}$$

This shows that

$$\eta_*(\mathfrak{G}) \propto (-1)^{\frac{m(m+1)}{2}} \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \tilde{\Theta}_{f^{(k)}, \varepsilon_{f^{(k)}} + \mu_{f,k}}^k \right) \cdot (\mathfrak{G}_{\mathbb{X}} \otimes \mathfrak{G}_{\mathbb{Y}}).$$

Hence,

$$\epsilon_* \circ \eta_*(\mathfrak{G}) \propto (-1)^{\frac{m(m+1)}{2}} \epsilon_* \left(\sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \tilde{\Theta}_{f^{(k)}, \varepsilon_{f^{(k)}} + \mu_{f,k}}^k \right) \cdot \mathfrak{G}_{\mathbb{Y}} \right) \cdot \mathfrak{G},$$

where $\epsilon : C(\tilde{\Gamma}_1) \rightarrow C(\tilde{\Gamma})$ is the morphism associated to the annihilation of the circle marked by \mathbb{Y} .

Since η is homogeneous of degree $m(N-m)$, the polynomial

$$\tilde{\Xi} = \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \tilde{\Theta}_{f^{(k)}, \varepsilon_{f^{(k)}} + \mu_{f,k}}^k \right)$$

is homogeneous of degree $2m(N-m)$. Let $\tilde{\Xi}^+$ be the part of $\tilde{\Xi}$ with positive total degree in \mathbb{X} . Then the total degree of $\tilde{\Xi}^+$ in \mathbb{Y} is less than $2m(N-m)$. By the definition of η , we know that $\epsilon_*(\tilde{\Xi}^+ \cdot \mathfrak{G}_{\mathbb{Y}}) = 0$. So

$$\epsilon_*(\tilde{\Xi} \cdot \mathfrak{G}_{\mathbb{Y}}) = \epsilon_*(\tilde{\Xi} - \tilde{\Xi}^+) \cdot \mathfrak{G}_{\mathbb{Y}} = \epsilon_*((\tilde{\Xi}|_{X_1=X_2=\dots=X_m=0}) \cdot \mathfrak{G}_{\mathbb{Y}}).$$

Next, consider $\hat{\Xi} := \tilde{\Xi}|_{X_1=X_2=\dots=X_m=0}$. Let $\hat{\Theta}_{j,\varepsilon}^k = \tilde{\Theta}_{j,\varepsilon}^k|_{X_1=X_2=\dots=X_m=0}$. Then

$$\hat{\Xi} = \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}}} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f^{(k)}, \varepsilon_{f^{(k)}} + \mu_{f,k}}^k \right).$$

Moreover, by equations (7.32), (7.33), (7.34), the definition of U_j in Lemma 7.38 and Lemma 4.1, we have, for $1 \leq k, j \leq m$,

$$\begin{aligned}\hat{\Theta}_{j,0}^k &= \hat{\Theta}_{j,0}^{m+k} = \begin{cases} -1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \\ \hat{\Theta}_{j,1}^k &= -\frac{\partial}{\partial X_k} U_j(\mathbb{X}, \mathbb{Y})|_{X_1=X_2=\dots=X_m=0} = (-1)^{k+j+1} j \cdot h_{N+1-k-j}(\mathbb{Y}), \\ \hat{\Theta}_{j,1}^{m+k} &= \frac{\partial}{\partial Y_k} U_j(\mathbb{X}, \mathbb{Y})|_{X_1=X_2=\dots=X_m=0} = (-1)^{k+j} (N+1-j) \cdot h_{N+1-k-j}(\mathbb{Y}).\end{aligned}$$

Now split $\hat{\Xi}$ into $\hat{\Xi} = \hat{\Xi}_1 + \hat{\Xi}_2$, where

$$\begin{aligned}\hat{\Xi}_1 &= \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_2} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right), \\ \hat{\Xi}_2 &= \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}} \setminus \mathcal{F}_2} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right).\end{aligned}$$

We compute $\hat{\Xi}_1$ first. For every pair of $f \in \mathcal{F}_2$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in I^m$, there is a bijection

$$f_\varepsilon : \{1, 2, \dots, 2m\} \rightarrow \{1, 2, \dots, m\} \times \mathbb{Z}_2$$

given by $f_\varepsilon(k) = (f(k), \varepsilon_{f(k)} + \mu_{f,k})$. Note that $(f, \varepsilon) \mapsto f_\varepsilon$ is a bijection from $\mathcal{F}_2 \times I^m$ to the set of bijections $\{1, 2, \dots, 2m\} \rightarrow \{1, 2, \dots, m\} \times \mathbb{Z}_2$. Define an order on $\{1, 2, \dots, m\} \times \mathbb{Z}_2$ by

$$(1, 1) < (1, 0) < (2, 1) < (2, 0) < \dots < (m, 1) < (m, 0).$$

Then, for $(f, \varepsilon) \in \mathcal{F}_2 \times I^m$,

$$|\varepsilon| + \nu_f = \#\{(k, k') \mid 1 \leq k < k' \leq 2m, f_\varepsilon(k) > f_\varepsilon(k')\}.$$

Thus,

$$\begin{aligned}\hat{\Xi}_1 &= \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_2} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right) \\ &= \begin{vmatrix} \hat{\Theta}_{1,1}^1 & \hat{\Theta}_{1,0}^1 & \hat{\Theta}_{2,1}^1 & \hat{\Theta}_{2,0}^1 & \dots & \hat{\Theta}_{m,1}^1 & \hat{\Theta}_{m,0}^1 \\ \hat{\Theta}_{1,1}^2 & \hat{\Theta}_{1,0}^2 & \hat{\Theta}_{2,1}^2 & \hat{\Theta}_{2,0}^2 & \dots & \hat{\Theta}_{m,1}^2 & \hat{\Theta}_{m,0}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\Theta}_{1,1}^{2m-1} & \hat{\Theta}_{1,0}^{2m-1} & \hat{\Theta}_{2,1}^{2m-1} & \hat{\Theta}_{2,0}^{2m-1} & \dots & \hat{\Theta}_{m,1}^{2m-1} & \hat{\Theta}_{m,0}^{2m-1} \\ \hat{\Theta}_{1,1}^{2m} & \hat{\Theta}_{1,0}^{2m} & \hat{\Theta}_{2,1}^{2m} & \hat{\Theta}_{2,0}^{2m} & \dots & \hat{\Theta}_{m,1}^{2m} & \hat{\Theta}_{m,0}^{2m} \end{vmatrix} \\ &= (-1)^{\frac{m(m+1)}{2}} \begin{vmatrix} \hat{\Theta}_{1,0}^1 & \dots & \hat{\Theta}_{m,0}^1 & \hat{\Theta}_{1,1}^1 & \dots & \hat{\Theta}_{m,1}^1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\Theta}_{1,0}^m & \dots & \hat{\Theta}_{m,0}^m & \hat{\Theta}_{1,1}^m & \dots & \hat{\Theta}_{m,1}^m \\ \hat{\Theta}_{1,0}^{m+1} & \dots & \hat{\Theta}_{m,0}^{m+1} & \hat{\Theta}_{1,1}^{m+1} & \dots & \hat{\Theta}_{m,1}^{m+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\Theta}_{1,0}^{2m} & \dots & \hat{\Theta}_{m,0}^{2m} & \hat{\Theta}_{1,1}^{2m} & \dots & \hat{\Theta}_{m,1}^{2m} \end{vmatrix}\end{aligned}$$

Note that both $m \times m$ blocks on the left are $-\mathbf{I}$, where \mathbf{I} is the $m \times m$ unit matrix. So

$$\begin{aligned} \hat{\Xi}_1 &= (-1)^{\frac{m(m+1)}{2}} \cdot (-1)^m \begin{vmatrix} \hat{\Theta}_{1,1}^{m+1} - \hat{\Theta}_{1,1}^1 & \cdots & \hat{\Theta}_{m,1}^{m+1} - \hat{\Theta}_{m,1}^1 \\ \cdots & \cdots & \cdots \\ \hat{\Theta}_{1,1}^{2m} - \hat{\Theta}_{1,1}^m & \cdots & \hat{\Theta}_{m,1}^{2m} - \hat{\Theta}_{m,1}^m \end{vmatrix} \\ &= (-1)^{\frac{m(m-1)}{2}} \det((-1)^{k+j} (N+1) h_{N+1-k-j}(\mathbb{Y}))_{1 \leq k, j \leq m} \\ &= (-1)^{\frac{m(m-1)}{2}} (N+1)^m \det(h_{N+1-k-j}(\mathbb{Y}))_{1 \leq k, j \leq m} \\ &= (N+1)^m \det(h_{N-m-k+j}(\mathbb{Y}))_{1 \leq k, j \leq m} \\ &= (N+1)^m S_{\lambda_{m, N-m}}(\mathbb{Y}), \end{aligned}$$

where $\lambda_{m, N-m} = \underbrace{(N-m \geq \cdots \geq N-m)}_{m \text{ parts}}$.

The sum $\hat{\Xi}_2$ is harder to understand. But, to determine $\epsilon_*(\hat{\Xi}_2 \cdot \mathfrak{G}_{\mathbb{Y}})$, we only need to know the coefficient of $S_{\lambda_{m, N-m}}(\mathbb{Y})$ in the decomposition of $\hat{\Xi}_2$ into Schur polynomials, which is not very hard to do. First, we consider the decomposition of $\hat{\Xi}_2$ into complete symmetric polynomials. Since $\hat{\Xi}_2$ is homogeneous of degree $2m(N-m)$, we have

$$\hat{\Xi}_2 = \sum_{|\lambda|=m(N-m), l(\lambda) \leq m} c_{\lambda} \cdot h_{\lambda}(\mathbb{Y}),$$

where $c_{\lambda} \in \mathbb{C}$. Note that $\hat{\Xi}_2$ is defined by

$$\hat{\Xi}_2 = \sum_{\varepsilon \in I^m} \sum_{f \in \mathcal{F}_{\text{even}} \setminus \mathcal{F}_2} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right),$$

in which every term is a scalar multiple of a complete symmetric polynomial associated to a partition of length $\leq m$. If the term corresponding to $\varepsilon \in I^m$ and $f \in \mathcal{F}_{\text{even}} \setminus \mathcal{F}_2$ makes a non-zero contribution to $c_{\lambda_{m, N-m}}$, then we know that, for every $k = 1, \dots, m$,

$$f(k) = \begin{cases} k & \text{if } \varepsilon_{f(k)} + \mu_{f,k} = 0, \\ m+1-k & \text{if } \varepsilon_{f(k)} + \mu_{f,k} = 1, \end{cases}$$

and

$$f(m+k) = \begin{cases} k & \text{if } \varepsilon_{f(m+k)} + \mu_{f,m+k} = 0, \\ m+1-k & \text{if } \varepsilon_{f(m+k)} + \mu_{f,m+k} = 1. \end{cases}$$

In particular,

$$f \in \mathcal{F}^{\diamond} := \{g \in \mathcal{F}_{\text{even}} \setminus \mathcal{F}_2 \mid g(k), g(m+k) \in \{k, m+1-k\}, \forall k = 1, \dots, m\}.$$

Now, for an $f \in \mathcal{F}^{\diamond}$, we have $f \in \mathcal{F}_{\text{even}} \setminus \mathcal{F}_2$. So there is a $j \in \{1, \dots, m\}$ such that $\#f^{-1}(j)$ is an even number greater than 2. From the above definition of \mathcal{F}^{\diamond} , we can see that $f^{-1}(j) \subset \{j, m+j, m+1-j, 2m+1-j\}$. Thus, $\#f^{-1}(j) = 4$ and $f^{-1}(j) = \{j, m+j, m+1-j, 2m+1-j\}$, which implies that $f^{-1}(m+1-j) = \emptyset$. Let $\varepsilon, \sigma \in I^m$ be such that $\varepsilon_{m+1-j} \neq \sigma_{m+1-j}$ and $\varepsilon_l = \sigma_l$ if $l \neq m+1-j$. Then

$$(-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right) = -(-1)^{|\sigma| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \sigma_{f(k)} + \mu_{f,k}}^k \right).$$

This implies that, for every $f \in \mathcal{F}^\diamond$,

$$\sum_{\varepsilon \in I^m} (-1)^{|\varepsilon| + \nu_f} \left(\prod_{k=1}^{2m} \hat{\Theta}_{f(k), \varepsilon_{f(k)} + \mu_{f,k}}^k \right) = 0.$$

Therefore, $c_{\lambda_{m,N-m}} = 0$. By Lemma 4.2, one can see that the coefficient of $S_{\lambda_{m,N-m}}(\mathbb{Y})$ is 0 in the decomposition of $\hat{\Xi}_2$ into Schur polynomials. So $\epsilon_*(\hat{\Xi}_2 \cdot \mathfrak{G}_{\mathbb{Y}}) = 0$.

Altogether, we have shown that

$$\begin{aligned} \epsilon_* \circ \eta_*(\mathfrak{G}) &\propto (-1)^{\frac{m(m+1)}{2}} \epsilon_*(\hat{\Xi} \cdot \mathfrak{G}_{\mathbb{Y}}) \cdot \mathfrak{G} \\ &= (-1)^{\frac{m(m+1)}{2}} \epsilon_*(\hat{\Xi}_1 \cdot \mathfrak{G}_{\mathbb{Y}}) \cdot \mathfrak{G} + (-1)^{\frac{m(m+1)}{2}} \epsilon_*(\hat{\Xi}_2 \cdot \mathfrak{G}_{\mathbb{Y}}) \cdot \mathfrak{G} \\ &= (-1)^{\frac{m(m+1)}{2}} (N+1)^m \epsilon_*(S_{\lambda_{m,N-m}}(\mathbb{Y}) \cdot \mathfrak{G}_{\mathbb{Y}}) \cdot \mathfrak{G} \\ &\propto (-1)^{\frac{m(m+1)}{2}} (N+1)^m \mathfrak{G} \neq 0, \end{aligned}$$

which proves the following lemma.

Lemma 7.40. *Let $\tilde{\Gamma}$ and $\tilde{\Gamma}_1$ be the colored MOY graphs in Figure 26, $\eta : C(\tilde{\Gamma}) \rightarrow C(\tilde{\Gamma}_1)$ the morphism associated to the saddle move and $\epsilon : C(\tilde{\Gamma}_1) \rightarrow C(\tilde{\Gamma})$ the morphism associated to the annihilation of the circle marked by \mathbb{Y} . Then $\epsilon_* \circ \eta_*(\mathfrak{G}) \propto \mathfrak{G}$, where \mathfrak{G} is the generating class of $H(\tilde{\Gamma})$. In particular, $\epsilon_* \circ \eta_* \neq 0$.*

Now, using an argument similar to that in the proof of Proposition 7.36, we can easily prove the following main conclusion of this subsection.

Proposition 7.41. *Let Γ and Γ_1 be the colored MOY graphs in Figure 24, $\eta : C(\Gamma) \rightarrow C(\Gamma_1)$ the morphism associated to the saddle move and $\epsilon : C(\Gamma_1) \rightarrow C(\Gamma)$ the morphism associated to circle annihilation. Then $\epsilon \circ \eta \approx \text{id}_{C(\Gamma)}$.*

Proof. We know that the subspace of $\text{Hom}_{HMF}(C(\Gamma), C(\Gamma))$ of elements of quantum degree 0 is 1-dimensional and spanned by $\text{id}_{C(\Gamma)}$. Note that the quantum degree of $\epsilon \circ \eta$ is 0. So, to prove that $\epsilon \circ \eta \approx \text{id}_{C(\Gamma)}$, we only need the fact that $\epsilon \circ \eta$ is not homotopic to 0, which follows from Lemma 7.40. \square

8. DIRECT SUM DECOMPOSITION (III)

In this section, we prove Theorem 8.1, which “categorifies” [29, Lemma 5.2] and generalizes direct sum decomposition (III) in [18].

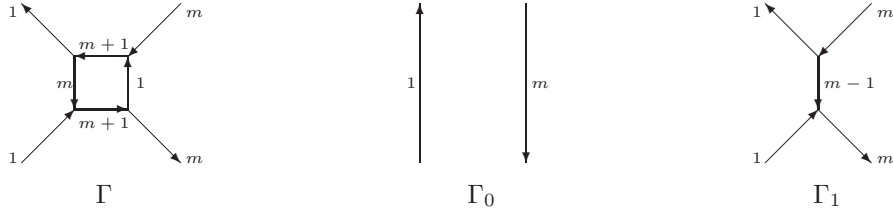


FIGURE 27.

Theorem 8.1. *Let Γ , Γ_0 and Γ_1 be the colored MOY graphs in Figure 27, where $m \leq N-1$. Then*

$$C(\Gamma) \simeq C(\Gamma_0) \oplus C(\Gamma_1)\{[N-m-1]\} \langle 1 \rangle.$$

Remark 8.2. Theorem 8.1 is not directly used in the proof of the invariance of the colored $\mathfrak{sl}(N)$ -homology.

8.1. Relating Γ and Γ_0 . In this subsection, we generalize the method in [41, Subsection 3.3] to construct morphisms between $C(\Gamma)$ and $C(\Gamma_0)$. In fact, the result we get in this subsection is slightly more general than what is needed to prove Theorem 8.1.

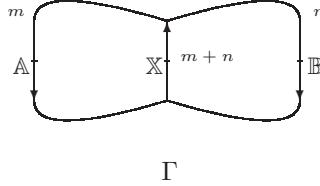


FIGURE 28.

Lemma 8.3. *Let Γ be the colored MOY graph in Figure 28. Then, as graded matrix factorizations over $\text{Sym}(\mathbb{A} \cup \mathbb{B})$,*

$$C(\Gamma) \simeq C(\emptyset) \otimes_{\mathbb{C}} (\text{Sym}(\mathbb{A} \cup \mathbb{B}) / (h_N(\mathbb{A} \cup \mathbb{B}), \dots, h_{N-m-n+1}(\mathbb{A} \cup \mathbb{B}))) \{q^{-(m+n)(N-m-n)}\} \langle m+n \rangle,$$

and, as graded \mathbb{C} -linear spaces,

$$\text{Hom}_{HMF}(C(\emptyset), C(\Gamma)) \cong \text{Hom}_{HMF}(C(\Gamma), C(\emptyset)) \cong H(\Gamma) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m+n \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix} \right\} \langle m+n \rangle.$$

In particular, the subspaces of these spaces consisting of homogeneous elements of quantum degree $-(m+n)(N-m-n) - mn$ are all 1-dimensional.

Proof. The first homotopy equivalence follows from Proposition 6.3 and the proof of Theorem 5.14. The rest of the lemma follows from this homotopy equivalence. \square

Denote by \bigcirc_{m+n} a circle colored by $m+n$. Then there are morphisms $C(\bigcirc_{m+n}) \xrightarrow{\phi} C(\Gamma)$ and $C(\Gamma) \xrightarrow{\bar{\phi}} C(\bigcirc_{m+n})$ induced by the edge splitting and merging. Denote by ι and ϵ the morphisms associated to creating and annihilating \bigcirc_{m+n} . Then $C(\emptyset) \xrightarrow{\tilde{\iota} := \phi \circ \iota} C(\Gamma)$ and $C(\Gamma) \xrightarrow{\tilde{\epsilon} := \epsilon \circ \bar{\phi}} C(\emptyset)$ are homogeneous morphisms of quantum degree $-(m+n)(N-m-n) - mn$ and \mathbb{Z}_2 -degree $m+n$.

Lemma 8.4. *$\tilde{\iota}$ and $\tilde{\epsilon}$ are not homotopic to 0, and, therefore, span the 1-dimensional subspaces of $\text{Hom}_{HMF}(C(\emptyset), C(\Gamma))$ and $\text{Hom}_{HMF}(C(\Gamma), C(\emptyset))$ of homogeneous elements of quantum degree $-(m+n)(N-m-n) - mn$.*

Proof. We only need to check that $\tilde{\iota}_*$ and $\tilde{\epsilon}_*$ are non-zero. By Lemma 8.3, as graded $\text{Sym}(\mathbb{A} \cup \mathbb{B})$ -modules,

$$H(\Gamma) \cong (\text{Sym}(\mathbb{A} \cup \mathbb{B}) / (h_N(\mathbb{A} \cup \mathbb{B}), \dots, h_{N-m-n+1}(\mathbb{A} \cup \mathbb{B}))) \{q^{-(m+n)(N-m-n)}\} \langle m+n \rangle.$$

We identify these two modules by this isomorphism. By the explicit descriptions of ι and ϕ in Section 7, it is easy to see that

$$\tilde{\iota}_*(1) = \phi_* \circ \iota_*(1) \propto \phi_*(1) \propto 1.$$

So $\tilde{\iota}_* \neq 0$.

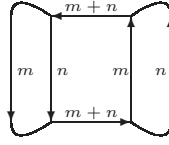
By the module structure of $H(\Gamma)$ and Theorems 4.3, 4.5, one can see that

$$\{S_\lambda(\mathbb{A}) \cdot S_\mu(\mathbb{A} \cup \mathbb{B}) \mid \lambda \leq \lambda_{m,n}, \mu \leq \lambda_{m+n, N-m-n}\}$$

is a \mathbb{C} -linear basis for $H(\Gamma)$. Using the explicit descriptions of ϵ and $\bar{\phi}$ in Section 7, it is easy to see that

$$\begin{aligned} \tilde{\epsilon}_*(S_{\lambda_{m,n}}(\mathbb{A}) \cdot S_{\lambda_{m+n, N-m-n}}(\mathbb{A} \cup \mathbb{B})) &= \epsilon_* \circ \bar{\phi}_*(S_{\lambda_{m,n}}(\mathbb{A}) \cdot S_{\lambda_{m+n, N-m-n}}(\mathbb{A} \cup \mathbb{B})) \\ &\propto \epsilon_*(S_{\lambda_{m+n, N-m-n}}(\mathbb{A} \cup \mathbb{B})) \propto 1. \end{aligned}$$

So $\tilde{\epsilon}_* \neq 0$. □



Γ_2

FIGURE 29.

Lemma 8.5. Denote by Γ_2 the colored MOY graph in Figure 29 and by \bigcirc_{m+n} a circle colored by $m+n$. As \mathbb{C} -linear spaces,

$$\mathrm{Hom}_{HMF}(C(\emptyset), C(\Gamma_2)) \cong \mathrm{Hom}_{HMF}(C(\Gamma_2), C(\emptyset)) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m+n \end{bmatrix} \cdot \begin{bmatrix} m+n \\ n \end{bmatrix}^2 \right\} \langle m+n \rangle.$$

In particular, the subspaces of these spaces consisting of homogeneous elements of quantum degree $-(m+n)(N-m-n)-2mn$ are 1-dimensional, and are spanned by

$$C(\emptyset) \xrightarrow{\iota} C(\bigcirc_{m+n}) \xrightarrow{\phi_1 \otimes \phi_2} C(\Gamma_2) \quad \text{and} \quad C(\Gamma_2) \xrightarrow{\bar{\phi}_1 \otimes \bar{\phi}_2} C(\bigcirc_{m+n}) \xrightarrow{\epsilon} C(\emptyset),$$

where ι and ϵ are morphisms associated to creating and annihilating \bigcirc_{m+n} , and ϕ_1, ϕ_2 (resp. $\bar{\phi}_1, \bar{\phi}_2$) are morphisms associated to the two apparent edge splitting (resp. merging.)

Proof. By Theorem 5.14 and Proposition 6.1, we have

$$C(\Gamma_2) \simeq C(\emptyset) \left\{ \begin{bmatrix} N \\ m+n \end{bmatrix} \cdot \begin{bmatrix} m+n \\ n \end{bmatrix}^2 \right\} \langle m+n \rangle.$$

The conclusion about $\mathrm{Hom}_{HMF}(C(\emptyset), C(\Gamma_2))$ and $\mathrm{Hom}_{HMF}(C(\Gamma_2), C(\emptyset))$ follows easily from this. It then follows that the subspaces of these spaces consisting of homogeneous elements of quantum degree $-(m+n)(N-m-n)-2mn$ are 1-dimensional. It is easy to check that $(\phi_1 \otimes \phi_2) \circ \iota$ and $\epsilon \circ (\bar{\phi}_1 \otimes \bar{\phi}_2)$ are both homogeneous with quantum degree $-(m+n)(N-m-n)-2mn$. So, to finish the proof, we only need to show that they are not null homotopic. It is clear that $(\phi_1 \otimes \phi_2)_*$ and ι_* are both injective, and ϵ_* and $(\bar{\phi}_1 \otimes \bar{\phi}_2)_*$ are both surjective. So $(\phi_1 \otimes \phi_2)_* \circ \iota_*$ and $\epsilon_* \circ (\bar{\phi}_1 \otimes \bar{\phi}_2)_*$ are both non-zero. Thus, $(\phi_1 \otimes \phi_2) \circ \iota$ and $\epsilon \circ (\bar{\phi}_1 \otimes \bar{\phi}_2)$ are not null homotopic. □

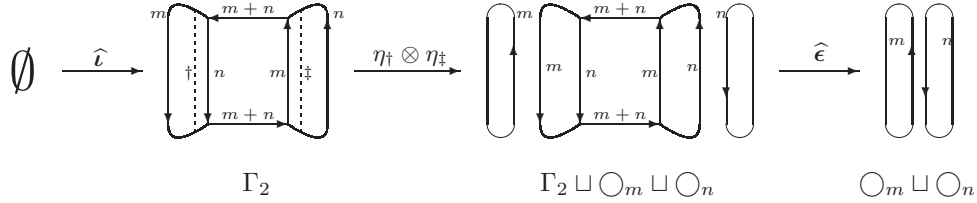


FIGURE 30.

For simplicity, we denote by $\widehat{\iota}$ the composition $C(\emptyset) \xrightarrow{(\phi_1 \otimes \phi_2) \circ \iota} C(\Gamma_2)$, and by $\widehat{\epsilon}$ the composition $C(\Gamma_2) \xrightarrow{\epsilon \circ (\overline{\phi}_1 \otimes \overline{\phi}_2)} C(\emptyset)$ in the rest of this section.

Lemma 8.6. *Denote by $\bigcirc_m \sqcup \bigcirc_n$ the disjoint union of two circles colored by m and n . Define the morphism $f : C(\emptyset) \rightarrow C(\bigcirc_m \sqcup \bigcirc_n)$ to be the composition in Figure 30, i.e. $f = \widehat{\epsilon} \circ (\eta_{\dagger} \otimes \eta_{\ddagger}) \circ \widehat{\iota}$. Then $f \approx \iota_m \otimes \iota_n$, where ι_m, ι_n are the morphisms associated to creating the two circles in $\bigcirc_m \sqcup \bigcirc_n$.*

Proof. It is easy to see that

$$\mathrm{Hom}_{HMF}(C(\emptyset), C(\bigcirc_m \sqcup \bigcirc_n)) \cong H(\bigcirc_m \sqcup \bigcirc_n) \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \cdot \begin{bmatrix} N \\ n \end{bmatrix} \right\} \langle m+n \rangle.$$

In particular, the subspace of $\mathrm{Hom}_{HMF}(C(\emptyset), C(\bigcirc_m \sqcup \bigcirc_n))$ of homogeneous elements of quantum degree $-m(N-m) - n(N-n)$ is 1-dimensional and spanned by $\iota_m \otimes \iota_n$. One can easily check that f is homogeneous of quantum degree $-m(N-m) - n(N-n)$. So, to prove the lemma, we only need to check that f is not null homotopic. We do this by showing that $f_*(1) \neq 0$.

Note that $f = \widehat{\epsilon} \circ (\eta_{\dagger} \otimes \eta_{\ddagger}) \circ \widehat{\iota} = (\widehat{\epsilon} \circ \eta_{\ddagger}) \circ (\eta_{\dagger} \circ \widehat{\iota})$.

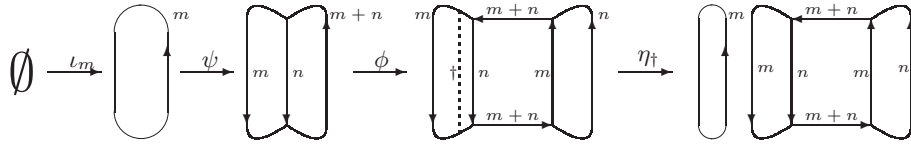


FIGURE 31.

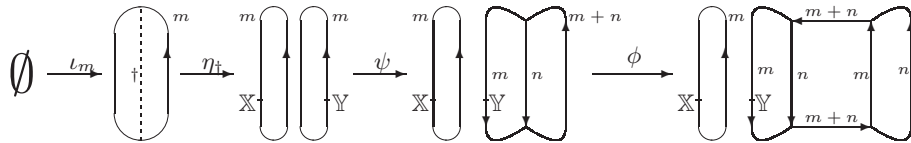


FIGURE 32.

We consider $\eta_{\dagger} \circ \widehat{\iota}$ first. By Proposition 7.32, one can see $\widehat{\iota} \approx \phi \circ \psi \circ \iota_m$, where the morphisms on the right hand side are given in Figure 31. So $\eta_{\dagger} \circ \widehat{\iota}$ is given by the composition in Figure 31. If we choose marked points appropriately, then $\phi \circ \psi$ and η_{\dagger} act on different factors in the tensor product of matrix factorizations.

So they commute. Thus $\eta_{\dagger} \circ \widehat{\iota} = (\phi \circ \psi) \circ (\eta_{\dagger} \circ \iota_m)$, where the composition on the right hand side is given in Figure 32. Denote by $\iota_{\mathbb{X}}$ and $\epsilon_{\mathbb{X}}$ (resp. $\iota_{\mathbb{Y}}$ and $\epsilon_{\mathbb{Y}}$) the morphisms associated to creating and annihilating the circle marked by \mathbb{X} (resp. \mathbb{Y} .) Then $(\iota_{\mathbb{X}})_*(1)$ and $(\iota_{\mathbb{Y}})_*(1)$ are the generating classes of the homology of the circles marked by \mathbb{X} and \mathbb{Y} . By Proposition 7.41, we have $\epsilon_{\mathbb{Y}} \circ \eta_{\dagger} \circ \iota_m \approx \iota_{\mathbb{X}}$. So $(\epsilon_{\mathbb{Y}} \circ \eta_{\dagger} \circ \iota_m)_*(1) \propto (\iota_{\mathbb{X}})_*(1)$. This implies that

$$(\eta_{\dagger} \circ \iota_m)_*(1) \propto (S_{\lambda_{m,N-m}}(\mathbb{Y}) + H) \cdot (\iota_{\mathbb{X}})_*(1) \otimes (\iota_{\mathbb{Y}})_*(1),$$

where H is an element in $\text{Sym}(\mathbb{X}|\mathbb{Y})$ whose total degree in \mathbb{Y} is less than $2m(N-m)$. By Proposition 7.32 and the definition of $\widehat{\iota}$, we have that

$$(\phi \circ \psi)_*((\iota_{\mathbb{Y}})_*(1)) \propto \widehat{\iota}_*(1).$$

Thus,

$$(\eta_{\dagger} \circ \widehat{\iota})_*(1) \propto (S_{\lambda_{m,N-m}}(\mathbb{Y}) + H) \cdot (\iota_{\mathbb{X}})_*(1) \otimes \widehat{\iota}_*(1).$$

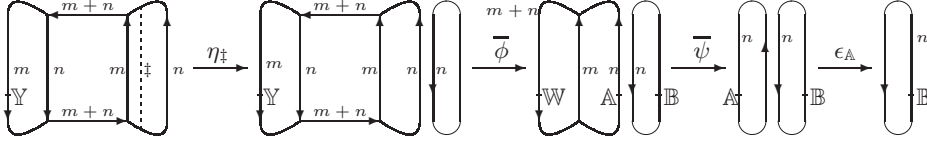


FIGURE 33.

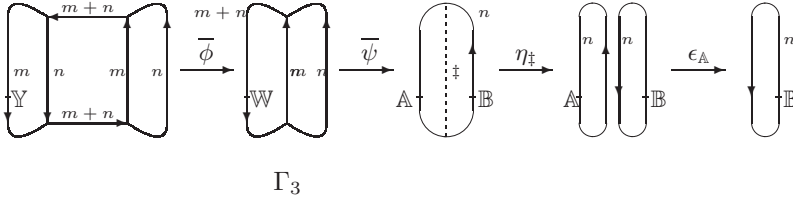


FIGURE 34.

Next we consider $\widehat{\epsilon} \circ \eta_{\dagger}$. Since the circle marked by \mathbb{X} is not affected by these morphisms, we temporarily drop that circle from our figures. By Proposition 7.32, $\widehat{\epsilon} \approx \epsilon_{\mathbb{A}} \circ \bar{\psi} \circ \bar{\phi}$, where the morphisms on the right hand side are given in Figure 33. So $\widehat{\epsilon} \circ \eta_{\dagger}$ is given by the composition in Figure 33. If we choose marked points appropriately, then $\bar{\psi} \circ \bar{\phi}$ and η_{\dagger} act on different factors in the tensor product of matrix factorizations. So they commute. Therefore, $\widehat{\epsilon} \circ \eta_{\dagger}$ is also given by the composition in Figure 34. By Proposition 7.41, $\epsilon_{\mathbb{A}} \circ \eta_{\dagger} \approx \text{id}$. So $\widehat{\epsilon} \circ \eta_{\dagger} \approx \bar{\psi} \circ \bar{\phi}$, where $\bar{\psi}$ and $\bar{\phi}$ are given in Figure 34. Denote by $\tilde{\iota}$ the morphism given in Lemma 8.4 associated to creating Γ_3 in Figure 34. Then, by [24, Proposition Gr3],

$$\bar{\phi}((S_{\lambda_{m,N-m}}(\mathbb{Y}) + H) \cdot (\iota_{\mathbb{X}})_*(1) \otimes \widehat{\iota}_*(1)) \propto (S_{\lambda_{m,N-m-n}}(\mathbb{W}) + h) \cdot (\iota_{\mathbb{X}})_*(1) \otimes \tilde{\iota}_*(1),$$

where h is an element of $\text{Sym}(\mathbb{X}|\mathbb{W})$ with total degree in \mathbb{W} less than $2m(N-m-n)$. Let $\psi_n : C(\bigcirc_n) \rightarrow C(\Gamma_3)$ be the morphism associated to the loop addition. (Note that this morphism is different from the ψ in Figures 31, 32.) Then, by Proposition 7.32, one can see that $\tilde{\iota} \approx \psi_n \circ \iota_n$.

Altogether, we have

$$\begin{aligned}
f_*(1) &\propto (\widehat{\epsilon} \circ \eta_{\dagger})_* \circ (\eta_{\dagger} \circ \widehat{\iota})_*(1) \\
&\propto (\widehat{\epsilon} \circ \eta_{\dagger})_* ((S_{\lambda_{m,N-m}}(\mathbb{Y}) + H) \cdot (\iota_{\mathbb{X}})_*(1) \otimes \widehat{\iota}_*(1)) \\
&\propto (\overline{\psi} \circ \overline{\phi})_* ((S_{\lambda_{m,N-m}}(\mathbb{Y}) + H) \cdot (\iota_{\mathbb{X}})_*(1) \otimes \widehat{\iota}_*(1)) \\
&\propto \overline{\psi}_* ((S_{\lambda_{m,N-m-n}}(\mathbb{W}) + h) \cdot (\iota_{\mathbb{X}})_*(1) \otimes \widetilde{\iota}_*(1)) \\
&\propto (\iota_{\mathbb{X}})_*(1) \otimes (\overline{\psi}_* \circ \mathbf{m}(S_{\lambda_{m,N-m-n}}(\mathbb{W}) + h) \circ (\psi_n)_* \circ (\iota_n)_*(1)) \\
&\propto (\iota_{\mathbb{X}})_*(1) \otimes (\iota_n)_*(1),
\end{aligned}$$

where the last step follows from equation (7.27) in Proposition 7.32. It is clear that the circle marked by \mathbb{X} is the “ \bigcirc_m ” in $\bigcirc_m \sqcup \bigcirc_n$. So the above computation shows that $f_*(1) \propto (\iota_m)_*(1) \otimes (\iota_n)_*(1) \neq 0$. And the lemma follows. \square

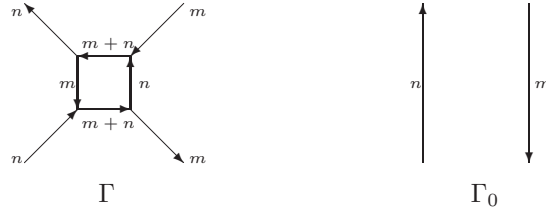


FIGURE 35.

Definition 8.7. Let Γ and Γ_0 be the colored MOY diagrams in Figure 35. (They are slightly more general than those in Theorem 8.1.) Define the morphism

$$F : C(\Gamma_0) \rightarrow C(\Gamma)$$

to be the composition in Figure 36, and the morphism

$$G : C(\Gamma) \rightarrow C(\Gamma_0)$$

to be the composition in Figure 37, where $\widehat{\iota}$, $\widehat{\epsilon}$ are as above, and η_{\square} , η_{\triangle} , η_{\dagger} , η_{\ddagger} are the morphisms associated to the corresponding saddle moves.

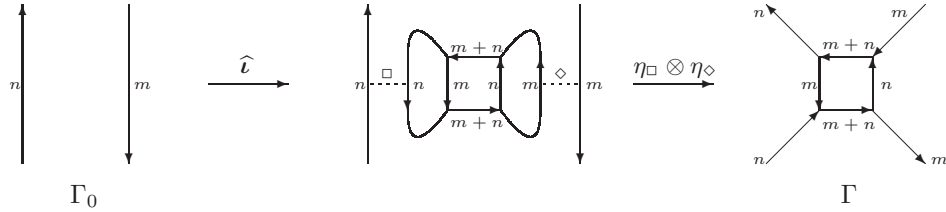
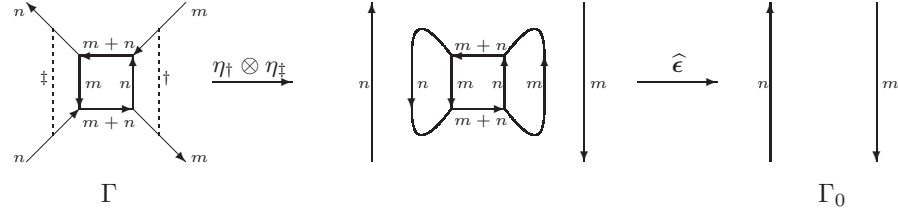


FIGURE 36. Definition of F

Proposition 8.8. Let F and G be the morphisms given in Definition 8.7. Then F and G are both homogeneous morphisms of quantum degree 0 and \mathbb{Z}_2 -degree 0. Moreover, $G \circ F \approx \text{id}_{C(\Gamma_0)}$.

FIGURE 37. Definition of G

Proof. Recall that $\widehat{\iota}$, $\widehat{\epsilon}$ are homogeneous morphisms of quantum degree $-(m+n)(N-m-n)-2mn$ and \mathbb{Z}_2 -degree $m+n$, and $\eta_{\square} \otimes \eta_{\diamond}$, $\eta_{\dagger} \otimes \eta_{\ddagger}$ are homogeneous morphisms of quantum degree $m(N-m)+n(N-n)$ and \mathbb{Z}_2 -degree $m+n$. So F and G are homogeneous morphisms of quantum degree 0 and \mathbb{Z}_2 -degree 0.

Next we consider the composition $G \circ F$. By marking the MOY graphs appropriately, $\eta_{\square} \otimes \eta_{\diamond}$ and G act on different factors of a tensor product. So they commute. Hence,

$$G \circ F = (\eta_{\square} \otimes \eta_{\diamond}) \circ G \circ \widehat{\iota} = (\eta_{\square} \otimes \eta_{\diamond}) \circ (\widehat{\epsilon} \circ (\eta_{\dagger} \otimes \eta_{\ddagger}) \circ \widehat{\iota}),$$

where the right hand side is the composition in Figure 38. By Lemma 8.6,

$$\widehat{\epsilon} \circ (\eta_{\dagger} \otimes \eta_{\ddagger}) \circ \widehat{\iota} \approx \iota_n \otimes \iota_m,$$

where ι_m and ι_n are the morphism associated to creating \bigcirc_m and \bigcirc_n . So

$$G \circ F \approx (\eta_{\square} \otimes \eta_{\diamond}) \circ (\iota_n \otimes \iota_m) = (\eta_{\square} \circ \iota_n) \otimes (\eta_{\diamond} \circ \iota_m) \approx \text{id}_{C(\Gamma_0)},$$

where the last step follows from Proposition 7.36. \square

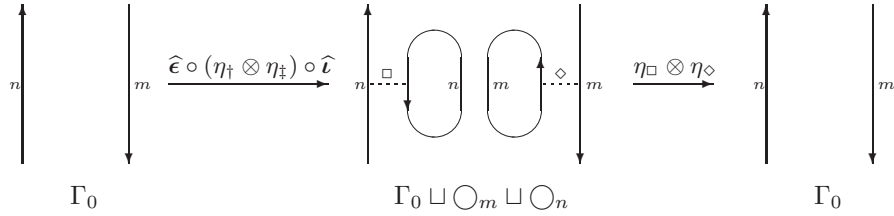


FIGURE 38.

8.2. Relating Γ and Γ_1 . Let Γ and Γ_1 be the colored MOY graphs in Figure 27. In this subsection, we generalize the method in [18, Section 6] to construct morphisms between $C(\Gamma)$ and $C(\Gamma_1)$. To do this, we need the following special case of Proposition 7.20.

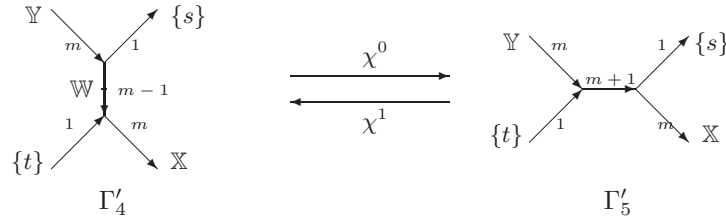


FIGURE 39.

Corollary 8.9. *Let Γ'_4 and Γ'_5 be the colored MOY graphs in Figure 39. Then there exist homogeneous morphisms*

$$\begin{aligned}\chi^0 &: C(\Gamma'_4) \rightarrow C(\Gamma'_5), \\ \chi^1 &: C(\Gamma'_5) \rightarrow C(\Gamma'_4),\end{aligned}$$

such that

- both χ^0 and χ^1 have quantum degree 1 and \mathbb{Z}_2 -degree 0,
- $\chi^1 \circ \chi^0 \simeq (s - t) \cdot \text{id}_{C(\Gamma'_4)}$ and $\chi^0 \circ \chi^1 \simeq (s - t) \cdot \text{id}_{C(\Gamma'_5)}$.

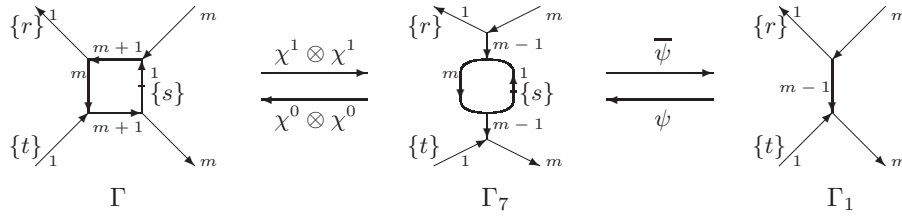


FIGURE 40.

If we cut Γ horizontally in half, then we get two copies of Γ'_5 in Figure 39. These correspond to two copies of Γ'_4 in Figure 39. Now we glue these two copies of Γ'_4 together along the original cutting points. This gives us Γ_7 in Figure 40. There are two χ^0 morphisms and two χ^1 morphisms corresponding to the two pairs of Γ'_4 and Γ'_5 . The morphism $\chi^0 \otimes \chi^0$ (resp. $\chi^1 \otimes \chi^1$) is the tensor product of these two χ^0 morphisms (resp. χ^1 morphisms.) Denote by $\psi : C(\Gamma_1) \rightarrow C(\Gamma_7)$ (resp. $\bar{\psi} : C(\Gamma_7) \rightarrow C(\Gamma_1)$) the morphism associated to the apparent loop addition (resp. removal.)

Definition 8.10. Define morphisms

$$\begin{aligned}\alpha &: C(\Gamma_1) \langle 1 \rangle \rightarrow C(\Gamma), \\ \beta &: C(\Gamma) \rightarrow C(\Gamma_1) \langle 1 \rangle\end{aligned}$$

by $\alpha = (\chi^0 \otimes \chi^0) \circ \psi$ and $\beta = \bar{\psi} \circ (\chi^1 \otimes \chi^1)$.

Moreover, for $j = 0, 1, \dots, N - m - 2$, define morphisms

$$\begin{aligned}\alpha_j &: C(\Gamma_1) \{q^{N-m-2-2j}\} \langle 1 \rangle \rightarrow C(\Gamma), \\ \beta_j &: C(\Gamma) \rightarrow C(\Gamma_1) \{q^{N-m-2-2j}\} \langle 1 \rangle\end{aligned}$$

by $\alpha_j = \mathbf{m}(s^{N-m-2-j}) \circ \alpha$ and $\beta_j = \beta \circ \mathbf{m}(h_j)$, where $\mathbf{m}(\bullet)$ is the morphism induced by multiplication of \bullet , and $h_j = h_j(\{r, s, t\})$ is the j -th complete symmetric polynomial in $\{r, s, t\}$.

Lemma 8.11. α_j and β_j are homogeneous morphisms that preserve both gradings. Moreover,

$$\beta_j \circ \alpha_i \approx \begin{cases} \text{id} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Proof. It is easy to verify the homogeneity and gradings of α_j and β_j . We omit it. Note that $\chi^0 \otimes \chi^0$ and $\chi^1 \otimes \chi^1$ are both $\mathbb{C}[r, s, t]$ -linear. So, by Corollary 8.9,

$$\begin{aligned}\beta_j \circ \alpha_i &= \bar{\psi} \circ (\chi^1 \otimes \chi^1) \circ \mathbf{m}(h_j) \circ \mathbf{m}(s^{N-m-2-i}) \circ (\chi^0 \otimes \chi^0) \circ \psi \\ &= \bar{\psi} \circ \mathbf{m}(h_j) \circ (\chi^1 \otimes \chi^1) \circ (\chi^0 \otimes \chi^0) \circ \mathbf{m}(s^{N-m-2-i}) \circ \psi \\ &\simeq \bar{\psi} \circ \mathbf{m}(h_j) \circ \mathbf{m}((r-s)(s-t)) \circ \mathbf{m}(s^{N-m-2-i}) \circ \psi.\end{aligned}$$

Denote by \hat{h}_j the j -th complete symmetric polynomial in $\{r, t\}$. Then

$$h_j = \sum_{l=0}^j s^l \hat{h}_{j-l}.$$

So

$$\begin{aligned}& s^{N-m-2-i}(r-s)(s-t)h_j \\ &= \sum_{l=0}^j s^{N-m-2-i+l}(-s^2 + (r+t)s - rt)\hat{h}_{j-l} \\ &= -\sum_{l=0}^j s^{N-m-i+l}\hat{h}_{j-l} + \sum_{l=-1}^{j-1} s^{N-m-i+l}(r+t)\hat{h}_{j-l-1} - \sum_{l=-2}^j s^{N-m-i+l}rt\hat{h}_{j-l-2} \\ &= -s^{N-m-i+j} + s^{N-m-i-1}\hat{h}_{j+1} - s^{N-m-i-2}rt\hat{h}_j \\ &\quad + \sum_{l=0}^{j-2} s^{N-m-i+l}(-\hat{h}_{j-l} + (r+t)\hat{h}_{j-l-1} - rt\hat{h}_{j-l-2}) \\ &= -s^{N-m-i+j} + s^{N-m-i-1}\hat{h}_{j+1} - s^{N-m-i-2}rt\hat{h}_j.\end{aligned}$$

Note that $\bar{\psi}$ is $\mathbb{C}[r, t]$ -linear. So

$$\begin{aligned}\beta_j \circ \alpha_i &\simeq \bar{\psi} \circ \mathbf{m}(h_j) \circ \mathbf{m}((r-s)(s-t)) \circ \mathbf{m}(s^{N-m-2-i}) \circ \psi \\ &= -\bar{\psi} \circ \mathbf{m}(s^{N-m-i+j}) \circ \psi + \mathbf{m}(\hat{h}_{j+1}) \circ \bar{\psi} \circ \mathbf{m}(s^{N-m-i-1}) \circ \psi \\ &\quad - \mathbf{m}(rt\hat{h}_j) \circ \bar{\psi} \circ \mathbf{m}(s^{N-m-i-2}) \circ \psi \\ &\approx \begin{cases} \text{id} & \text{if } i = j, \\ 0 & \text{if } i > j, \end{cases}\end{aligned}$$

where the last step follows from Proposition 7.32. \square

Proposition 8.12. *Let Γ and Γ_1 be as in Theorem 8.1. Then there exist homogeneous morphisms*

$$\begin{aligned}\vec{\alpha} &: C(\Gamma_1)\{[N-m-1]\}\langle 1 \rangle \rightarrow C(\Gamma), \\ \vec{\beta} &: C(\Gamma) \rightarrow C(\Gamma_1)\{[N-m-1]\}\langle 1 \rangle,\end{aligned}$$

that preserve both gradings and satisfy

$$\vec{\beta} \circ \vec{\alpha} \simeq \text{id}.$$

Proof. The β_j in Definition 8.10 is defined up to homotopy and scaling. From Lemma 8.11, we know

$$\beta_j \circ \alpha_i \approx \begin{cases} \text{id} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

So, by choosing an appropriate scalar for each β_j , we can make

$$(8.1) \quad \beta_j \circ \alpha_i \simeq \begin{cases} \text{id} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

We assume (8.1) is true in the rest of this proof.

Define $\tau_{j,i} : C(\Gamma_1)\{q^{N-m-2-2i}\} \langle 1 \rangle \rightarrow C(\Gamma_1)\{q^{N-m-2-2j}\} \langle 1 \rangle$ by

$$\tau_{j,i} = \begin{cases} \sum_{l \geq 1} \sum_{i < k_1 < \dots < k_{l-1} < j} (-1)^l (\beta_j \circ \alpha_{k_{l-1}}) \circ (\beta_{k_{l-1}} \circ \alpha_{k_{l-2}}) \circ \dots \circ (\beta_{k_1} \circ \alpha_i) & \text{if } i < j \\ \text{id} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Then define $\hat{\beta}_j : C(\Gamma) \rightarrow C(\Gamma_1)\{q^{N-m-2-2j}\} \langle 1 \rangle$ by

$$\hat{\beta}_j = \sum_{k=0}^{N-m-2} \tau_{j,k} \circ \beta_k.$$

Note that

$$C(\Gamma_1)\{[N-m-1]\} \langle 1 \rangle \cong \bigoplus_{j=0}^{N-m-2} C(\Gamma_1)\{q^{N-m-2-2j}\} \langle 1 \rangle.$$

We define $\vec{\alpha} : C(\Gamma_1)\{[N-m-1]\} \langle 1 \rangle \rightarrow C(\Gamma)$ by

$$\vec{\alpha} = (\alpha_0, \dots, \alpha_{N-m-2}),$$

and define $\vec{\beta} : C(\Gamma) \rightarrow C(\Gamma_1)\{[N-m-1]\} \langle 1 \rangle$ by

$$\vec{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \vdots \\ \hat{\beta}_{N-m-2} \end{pmatrix}$$

It is easy to check that α_i and $\hat{\beta}_j$ are homogeneous morphisms preserving both gradings. So are $\vec{\alpha}$ and $\vec{\beta}$.

Next we prove that $\vec{\beta} \circ \vec{\alpha} \simeq \text{id}$. Consider

$$\hat{\beta}_j \circ \alpha_i = \sum_{k=0}^{N-m-2} \tau_{j,k} \circ (\beta_k \circ \alpha_i).$$

By (8.1) and the definition of $\tau_{j,k}$, it is easy to see that

$$\hat{\beta}_j \circ \alpha_i \simeq \begin{cases} \text{id} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Now assume $i < j$. Again, by (8.1) and the definition of $\tau_{j,k}$, we have

$$\begin{aligned}
& \hat{\beta}_j \circ \alpha_i \\
&= \sum_{k=0}^{N-m-2} \tau_{j,k} \circ (\beta_k \circ \alpha_i) \\
&\simeq \sum_{k=i}^j \tau_{j,k} \circ (\beta_k \circ \alpha_i) \\
&= \tau_{j,i} \circ (\beta_i \circ \alpha_i) + \tau_{j,j} \circ (\beta_j \circ \alpha_i) + \sum_{i < k < j} \tau_{j,k} \circ (\beta_k \circ \alpha_i) \\
&\simeq \tau_{j,i} + \beta_j \circ \alpha_i \\
&\quad + \sum_{l \geq 1} \sum_{i < k < k_1 < \dots < k_{l-1} < j} (-1)^l (\beta_j \circ \alpha_{k_{l-1}}) \circ (\beta_{k_{l-1}} \circ \alpha_{k_{l-2}}) \circ \dots \circ (\beta_{k_1} \circ \alpha_k) \circ (\beta_k \circ \alpha_i) \\
&= \tau_{j,i} - \tau_{j,i} = 0
\end{aligned}$$

Altogether, we have $\vec{\beta} \circ \vec{\alpha} \simeq \text{id}$. \square

8.3. Proof of Theorem 8.1. With the morphisms constructed in the two preceding subsections, we are now ready to prove Theorem 8.1. Our method is a generalization of that in [18] and [41].

Lemma 8.13. *Let Γ , Γ_0 and Γ_1 be the colored MOY graphs in Figure 27. Suppose that F and G are the morphisms defined in Definition 8.7 (for $n = 1$), and $\vec{\alpha}$ and $\vec{\beta}$ are the morphisms given in Proposition 8.12. Then $\vec{\beta} \circ F \simeq 0$ and $G \circ \vec{\alpha} \simeq 0$.*

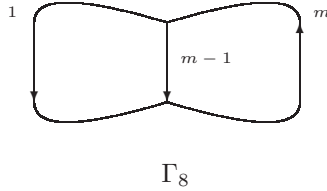


FIGURE 41.

Proof. Let Γ_8 be the colored MOY graph in Figure 41. Denote by $\bar{\Gamma}_0$ (resp. $\bar{\Gamma}_1$, $\bar{\Gamma}_8$) the colored MOY graph obtained by reversing the orientation of all edges of Γ_0 (resp. Γ_1 , Γ_8 .) Let \bigcirc_m be a circle colored by m . Then

$$\begin{aligned}
\text{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1)) &\cong H(C(\Gamma_1) \otimes C(\bar{\Gamma}_0)) \{q^{m(N-m)+N-1}\} \langle m+1 \rangle \\
&\cong H(\Gamma_8) \{q^{m(N-m)+N-1}\} \langle m+1 \rangle \\
&\cong H(\bigcirc_m) \{[m] \cdot q^{m(N-m)+N-1}\} \langle m+1 \rangle \\
&\cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \cdot [m] \cdot q^{m(N-m)+N-1} \right\} \langle 1 \rangle.
\end{aligned}$$

In particular, the lowest non-vanishing quantum grading of $\text{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1))$ is $N - m$. But when viewed as a morphism $C(\Gamma_0) \rightarrow C(\Gamma_1)$, the quantum degree of $\hat{\beta}_j \circ F$ is $-N + m + 2 + 2j$, which is less than $N - m$ for $j = 0, 1, \dots, N - m - 2$. So $\hat{\beta}_j \circ F \simeq 0$, for $j = 0, 1, \dots, N - m - 2$. That is $\vec{\beta} \circ F \simeq 0$.

Clearly, the matrix factorization of Γ_1 is the same as that of Γ'_4 in Figure 39. Similar Lemma 7.22, one can check that

$$C(\Gamma_1) \simeq M' := \left(\begin{array}{c} * \\ \dots \\ * \\ \dots \\ * \\ * \end{array} \begin{array}{l} (X_1 - Y_1) + (s - t) \\ \dots \\ (X_k - Y_k) + (s - t) \sum_{l=0}^{k-1} (-t)^{k-1-l} X_l \\ \dots \\ (X_m - Y_m) + (s - t) \sum_{l=0}^{m-1} (-t)^{m-1-l} X_l \\ \sum_{l=0}^m (-t)^{m-l} X_l \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\{s\}|\{t\})} \{q^{-m+1}\},$$

where \mathbb{X} , \mathbb{Y} , $\{s\}$, $\{t\}$ are markings of Γ'_4 in Figure 39. Mark the corresponding end points of Γ_0 by the same alphabets. Then

$$\begin{aligned} \text{Hom}_{HMF}(C(\Gamma_1), C(\Gamma_0)) &\cong \text{Hom}_{HMF}(M', C(\Gamma_0)) \\ &\cong H(C(\Gamma_0) \otimes M'_\bullet) \\ &\cong H(C(\Gamma_0) \otimes C(\overline{\Gamma}_1)) \{q^{m(N-m)+N-1}\} \langle m+1 \rangle \\ &\cong H(\overline{\Gamma}_8) \{q^{(m(N-m)+N-1)}\} \langle m+1 \rangle \\ &\cong H(\Gamma_8) \{q^{(m(N-m)+N-1)}\} \langle m+1 \rangle \\ &\cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m \end{bmatrix} \cdot [m] \cdot q^{m(N-m)+N-1} \right\} \langle 1 \rangle. \end{aligned}$$

In particular, the lowest non-vanishing quantum grading of $\text{Hom}_{HMF}(C(\Gamma_1), C(\Gamma_0))$ is $N - m$. But when viewed as a morphism $C(\Gamma_1) \rightarrow C(\Gamma_0)$, the quantum degree of $G \circ \alpha_j$ is $N - m - 2 - 2j$, which is less than $N - m$ for $j = 0, 1, \dots, N - m - 2$. So $G \circ \alpha_j \simeq 0$, for $j = 0, 1, \dots, N - m - 2$. That is $G \circ \vec{\alpha} \simeq 0$. \square

Recall that the morphisms F and G are defined only up to scaling and homotopy, and, by Proposition 8.8, we have $G \circ F \approx \text{id}_{C(\Gamma_0)}$. So, by choosing appropriate scalars, we can make

$$(8.2) \quad G \circ F \simeq \text{id}_{C(\Gamma_0)}.$$

For minor technical convenience, we assume that (8.2) is true for the rest of this section.

Lemma 8.14. *Let Γ , Γ_0 and Γ_1 be the colored MOY graphs in Figure 27. Then there exists a graded matrix factorization M , such that*

$$C(\Gamma) \simeq C(\Gamma_0) \oplus C(\Gamma_1) \{[N - m - 1]\} \langle 1 \rangle \oplus M.$$

Proof. Define morphisms

$$\begin{aligned} \tilde{F} &: C(\Gamma_0) \oplus C(\Gamma_1) \{[N - m - 1]\} \langle 1 \rangle \rightarrow C(\Gamma), \\ \tilde{G} &: C(\Gamma) \rightarrow C(\Gamma_0) \oplus C(\Gamma_1) \{[N - m - 1]\} \langle 1 \rangle \end{aligned}$$

by

$$\tilde{F} = (F, \vec{\alpha}) \quad \text{and} \quad \tilde{G} = \begin{pmatrix} G \\ \vec{\beta} \end{pmatrix}.$$

Then, by Proposition 8.8 (especially (8.2) above), Proposition 8.12 and Lemma 8.13, \tilde{F} and \tilde{G} are homogeneous morphisms preserving both gradings and satisfy

$$\tilde{G} \circ \tilde{F} \simeq \text{id}_{C(\Gamma_0) \oplus C(\Gamma_1) \{[N - m - 1]\} \langle 1 \rangle}.$$

Therefore, $\tilde{F} \circ \tilde{G} : C(\Gamma) \rightarrow C(\Gamma)$ preserves both gradings and satisfies

$$(\tilde{F} \circ \tilde{G}) \circ (\tilde{F} \circ \tilde{G}) \simeq \tilde{F} \circ \tilde{G}.$$

By Lemma 3.14, there exists a graded matrix factorization M such that

$$C(\Gamma) \simeq C(\Gamma_0) \oplus C(\Gamma_1)\{[N - m - 1]\} \langle 1 \rangle \oplus M.$$

□

Lemma 8.15. *Let M be as in Lemma 8.14. Then $M \simeq 0$.*

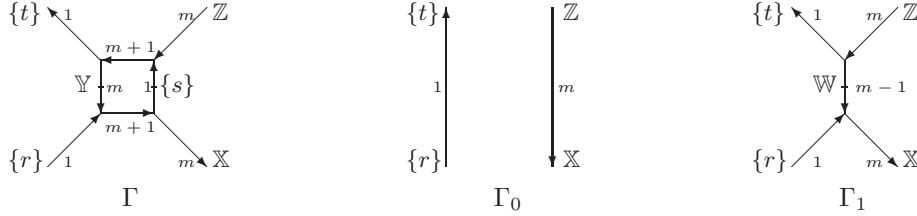


FIGURE 42.

Proof. Mark Γ , Γ_0 and Γ_1 as in Figure 42.

Consider homology of matrix factorizations with non-vanishing potentials defined in Definition 3.4. By Corollary 3.9, to show $M \simeq 0$, we only need to show that $H(M) = 0$, or, equivalently, $\text{gdim}(M) = 0$. But, by Lemma 8.14, we have

$$H(\Gamma) \cong H(\Gamma_0) \oplus H(\Gamma_1)\{[N - m - 1]\} \langle 1 \rangle \oplus H(M).$$

So,

$$\text{gdim}(C(\Gamma)) = \text{gdim}(C(\Gamma_0)) + \tau \cdot [N - m - 1] \cdot \text{gdim}(C(\Gamma_1)) + \text{gdim}(M).$$

Therefore, to prove the lemma, we only need to show that

$$(8.3) \quad \text{gdim}(C(\Gamma)) = \text{gdim}(C(\Gamma_0)) + \tau \cdot [N - m - 1] \cdot \text{gdim}(C(\Gamma_1))$$

In the rest of this proof, we prove (8.3) by directly computing $\text{gdim}(C(\Gamma))$, $\text{gdim}(C(\Gamma_0))$ and $\text{gdim}(C(\Gamma_1))$.

We compute $\text{gdim}(C(\Gamma))$ first. Let $\mathbb{A} = \mathbb{X} \cup \{s\}$, $\mathbb{B} = \mathbb{Y} \cup \{r\}$, $\mathbb{D} = \mathbb{Y} \cup \{t\}$, $\mathbb{E} = \mathbb{Z} \cup \{s\}$. By Lemma 5.12, we contract the two edges in Γ of color $m + 1$ and get

$$C(\Gamma) \simeq \begin{pmatrix} U_1 & A_1 - B_1 \\ \cdots & \cdots \\ U_{m+1} & A_{m+1} - B_{m+1} \\ V_1 & D_1 - E_1 \\ \cdots & \cdots \\ V_{m+1} & D_{m+1} - E_{m+1} \end{pmatrix}_{\text{Sym}(\mathbb{X}[\mathbb{Y}|\mathbb{Z}|\{r\}|\{s\}|\{t\}]}} \{q^{-2m}\},$$

where A_j is the j -th elementary symmetric function in \mathbb{A} and so on, and

$$\begin{aligned} U_j &= \frac{p_{m+1,N+1}(B_1, \dots, B_{j-1}, A_j, \dots, A_{m+1}) - p_{m+1,N+1}(B_1, \dots, B_j, A_{j+1}, \dots, A_{m+1})}{A_j - B_j}, \\ V_j &= \frac{p_{m+1,N+1}(E_1, \dots, E_{j-1}, D_j, \dots, D_{m+1}) - p_{m+1,N+1}(E_1, \dots, E_j, D_{j+1}, \dots, D_{m+1})}{D_j - E_j}. \end{aligned}$$

Recall that $C(\Gamma)$ is viewed as a matrix factorization over $\text{Sym}(\mathbb{X}|\mathbb{Z}|\{r\}|\{t\})$. So the corresponding maximal ideal for $C(\Gamma)$ is the ideal $\mathfrak{J} = (X_1, \dots, X_m, Z_1, \dots, Z_m, r, t)$ of $\text{Sym}(\mathbb{X}|\mathbb{Z}|\{r\}|\{t\})$. Identify

$$\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{Z}|\{r\}|\{s\}|\{t\})/\mathfrak{J} \cdot \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{Z}|\{r\}|\{s\}|\{t\}) = \text{Sym}(\mathbb{Y}|\{s\})$$

by the relations

$$(8.4) \quad X_1 = \dots = X_m = Z_1 = \dots = Z_m = r = t = 0.$$

Then

$$\begin{aligned} C(\Gamma)/\mathfrak{J} \cdot C(\Gamma) &\simeq \begin{pmatrix} U_1 & s - Y_1 \\ U_2 & -Y_2 \\ \dots & \dots \\ U_m & -Y_m \\ U_{m+1} & 0 \\ V_1 & Y_1 - s \\ V_2 & Y_2 \\ \dots & \dots \\ V_m & Y_m \\ V_{m+1} & 0 \end{pmatrix}_{\text{Sym}(\mathbb{Y}|\{s\})} \{q^{-2m}\} \\ &\simeq \begin{pmatrix} U_{m+1} & 0 \\ V_1 & 0 \\ V_2 & 0 \\ \dots & \dots \\ V_m & 0 \\ V_{m+1} & 0 \end{pmatrix}_{\mathbb{C}[s]} \{q^{-2m}\}, \end{aligned}$$

where, in the second step, we applied Proposition 2.19 repeatedly to eliminate the indeterminants Y_1, \dots, Y_m by the relations

$$(8.5) \quad Y_1 = s, \quad Y_2 = \dots = Y_m = 0.$$

Under relations (8.4) and (8.5), we have

$$A_j = B_j = D_j = E_j = \begin{cases} s & \text{if } j = 1, \\ 0 & \text{if } j = 2, \dots, m+1. \end{cases}$$

So, by Lemma 4.1, we have

$$\begin{aligned} U_j = V_j &= \frac{\partial p_{m+1, N+1}(A_1, \dots, A_{m+1})}{\partial A_j} \Big|_{A_1=s, A_2=\dots=A_{m+1}=0} \\ &= (-1)^j (N+1) h_{m+1, N+1-j}(s, 0, \dots, 0) \\ &= (-1)^j (N+1) s^{N+1-j}. \end{aligned}$$

Using Lemma 7.13 and Corollary 2.25, we then have

$$\begin{aligned}
H(\Gamma) &\cong H\left(\begin{pmatrix} s^{N-m} & 0 \\ s^N & 0 \\ s^{N-1} & 0 \\ \dots & \dots \\ s^{N-m} & 0 \end{pmatrix}\right)_{\mathbb{C}[s]} \{q^{-2m}\} \\
&\cong H\left(\begin{pmatrix} 0_N & 0 \\ 0_{N-1} & 0 \\ \dots & \dots \\ 0_{N-m} & 0 \end{pmatrix}\right)_{\mathbb{C}[s]/(s^{N-m})} \{q^{1-N}\} \langle 1 \rangle \\
&\cong \left(\begin{pmatrix} 0_N & 0 \\ 0_{N-1} & 0 \\ \dots & \dots \\ 0_{N-m} & 0 \end{pmatrix}\right)_{\mathbb{C}[s]/(s^{N-m})} \{q^{1-N}\} \langle 1 \rangle,
\end{aligned}$$

where 0_j is “a 0 that has degree $2j$ ”. So

$$\begin{aligned}
\text{gdim}(C(\Gamma)) &= \tau \cdot q^{1-N} \cdot \left(\sum_{k=0}^{N-m-1} q^{2k} \right) \cdot \prod_{j=1}^{m+1} (1 + \tau q^{2j-N-1}) \\
&= \tau \cdot q^{-m} \cdot [N-m] \cdot \prod_{j=1}^{m+1} (1 + \tau q^{2j-N-1}).
\end{aligned}$$

Next, we compute $\text{gdim}(C(\Gamma_0))$. Let

$$\begin{aligned}
\hat{U} &= \frac{t^{N+1} - r^{N+1}}{t - r} \\
\hat{U}_j &= \frac{p_{m,N+1}(Z_1, \dots, Z_{j-1}, X_j, \dots, X_m) - p_{m,N+1}(Z_1, \dots, Z_j, X_{j+1}, \dots, X_m)}{X_j - Z_j}
\end{aligned}$$

Then

$$C(\Gamma_0) = \left(\begin{pmatrix} \hat{U} & t - r \\ \hat{U}_1 & X_1 - Z_1 \\ \dots & \dots \\ \hat{U}_m & X_m - Z_m \end{pmatrix} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Z}|\{r\}|\{t\})}$$

So

$$C(\Gamma_0)/\mathfrak{I} \cdot C(\Gamma_0) \cong \left(\begin{pmatrix} 0_N & 0 \\ 0_N & 0 \\ 0_{N-1} & 0 \\ \dots & \dots \\ 0_{N-m+1} & 0 \end{pmatrix} \right)_{\mathbb{C}}$$

and

$$\text{gdim}(C(\Gamma_0)) = (1 + \tau q^{1-N}) \cdot \prod_{j=1}^m (1 + \tau q^{2j-N-1}).$$

Now we compute $\text{gdim}(C(\Gamma_1))$. Let $\mathbb{F} = \mathbb{W} \cup \{t\}$ and $\mathbb{G} = \mathbb{W} \cup \{r\}$. Define

$$\begin{aligned}\bar{U}_j &= \frac{p_{m,N+1}(Z_1, \dots, Z_{j-1}, F_j, \dots, F_m) - p_{m,N+1}(Z_1, \dots, Z_j, F_{j+1}, \dots, F_m)}{F_j - Z_j}, \\ \bar{V}_j &= \frac{p_{m,N+1}(G_1, \dots, G_{j-1}, X_j, \dots, X_m) - p_{m,N+1}(G_1, \dots, G_j, X_{j+1}, \dots, X_m)}{X_j - G_j}.\end{aligned}$$

Then

$$C(\Gamma_1) = \begin{pmatrix} \bar{U}_1 & F_1 - Z_1 \\ \dots & \dots \\ \bar{U}_m & F_m - Z_m \\ \bar{V}_1 & X_1 - G_1 \\ \dots & \dots \\ \bar{V}_m & X_m - G_m \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Z}|\mathbb{W}|\{r\}|\{t\})} \{q^{1-m}\}.$$

Identify

$$\text{Sym}(\mathbb{X}|\mathbb{Z}|\mathbb{W}|\{r\}|\{t\})/\mathfrak{I} \cdot \text{Sym}(\mathbb{X}|\mathbb{Z}|\mathbb{W}|\{r\}|\{t\}) = \text{Sym}(\mathbb{W})$$

by relations (8.4). Then, by Proposition 2.19,

$$\begin{aligned}C(\Gamma_1)/\mathfrak{I} \cdot C(\Gamma_1) &\cong \begin{pmatrix} \bar{U}_1 & W_1 \\ \dots & \dots \\ \bar{U}_{m-1} & W_{m-1} \\ \bar{U}_m & 0 \\ \bar{V}_1 & -W_1 \\ \dots & \dots \\ \bar{V}_{m-1} & -W_{m-1} \\ \bar{V}_m & 0 \end{pmatrix}_{\text{Sym}(\mathbb{W})} \{q^{1-m}\} \\ &\simeq \begin{pmatrix} 0_{N+1-m} & 0 \\ 0_N & 0 \\ 0_{N-1} & 0 \\ \dots & \dots \\ 0_{N-m+1} & 0 \end{pmatrix}_{\mathbb{C}} \{q^{1-m}\}.\end{aligned}$$

So

$$\text{gdim}(C(\Gamma_1)) = q^{1-m} \cdot (1 + \tau q^{2m-N-1}) \cdot \prod_{j=1}^m (1 + \tau q^{2j-N-1}).$$

Write

$$P = \prod_{j=1}^m (1 + \tau q^{2j-N-1}).$$

Then

$$\begin{aligned}\text{gdim}(C(\Gamma)) &= \tau \cdot q^{-m} \cdot [N-m] \cdot (1 + \tau q^{2m-N+1}) \cdot P, \\ \text{gdim}(C(\Gamma_0)) &= (1 + \tau q^{1-N}) \cdot P, \\ \text{gdim}(C(\Gamma_1)) &= q^{1-m} \cdot (1 + \tau q^{2m-N-1}) \cdot P.\end{aligned}$$

Note that

$$[N-m] = [N-m-1] \cdot q + q^{-(N-m-1)}.$$

So,

$$\begin{aligned}
& \text{gdim}(C(\Gamma)) - \text{gdim}(C(\Gamma_0)) - \tau \cdot [N - m - 1] \cdot \text{gdim}(C(\Gamma_1)) \\
&= ((\tau \cdot q^{-m} + q^{m-N+1}) \cdot [N - m] - 1 - \tau \cdot q^{1-N} - (q^{1-m} + \tau \cdot q^{m-N}) \cdot [N - m - 1]) \cdot P \\
&= ((\tau \cdot q^{-m} + q^{m-N+1}) \cdot ([N - m - 1] \cdot q + q^{-(N-m-1)}) - 1 - \tau \cdot q^{1-N} \\
&\quad - (q^{1-m} + \tau \cdot q^{m-N}) \cdot [N - m - 1]) \cdot P \\
&= ([N - m - 1] \cdot (q - q^{-1}) \cdot q^{m-N+1} + q^{2(m-N+1)} - 1) \cdot P \\
&= 0.
\end{aligned}$$

This shows that (8.3) is true. \square

Proof of Theorem 8.1. Lemmas 8.14 and 8.15 imply Theorem 8.1. \square

9. DIRECT SUM DECOMPOSITION (IV)

The main objective of this section is to prove Theorem 9.1, which “categorifies” [29, Lemma A.7] and generalizes direct sum decomposition (IV) in [18].

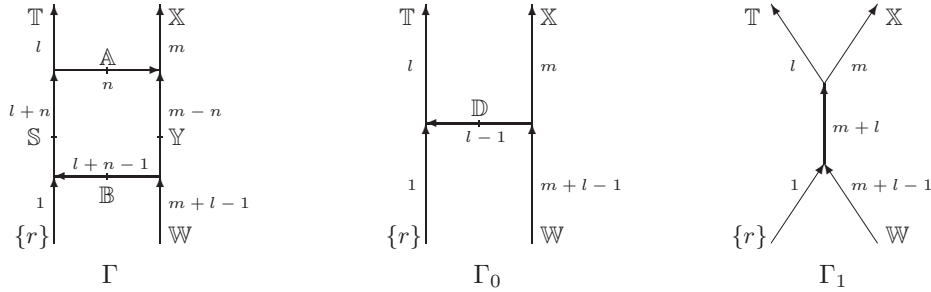


FIGURE 43.

Theorem 9.1. *Let Γ , Γ_0 and Γ_1 be the MOY graphs in Figure 43, where l, m, n are integers satisfying $0 \leq n \leq m \leq N$ and $0 \leq l, m + l - 1 \leq N$. Then*

$$(9.1) \quad C(\Gamma) \simeq C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\}.$$

Similarly, if $\bar{\Gamma}$, $\bar{\Gamma}_0$ and $\bar{\Gamma}_1$ are Γ , Γ_0 , Γ_1 with the orientation of every edge reversed, then

$$(9.2) \quad C(\bar{\Gamma}) \simeq C(\bar{\Gamma}_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\bar{\Gamma}_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\}.$$

The proofs of decompositions (9.1) and (9.2) are almost identical. So we only prove (9.1) in this paper and leave (9.2) to the readers.

Remark 9.2. Although direct sum decomposition (IV) is formulated in a different form in [18], its proof there comes down to establishing the decomposition

$$(9.3) \quad C(\Gamma') \simeq C(\Gamma'_0) \oplus C(\Gamma'_1),$$

where Γ' , Γ'_0 and Γ'_1 are given in Figure 44. This is also what is actually used in the proof of the invariance of the Khovanov-Rozansky $\mathfrak{sl}(N)$ -homology under Reidemeister move III. Clearly, if we specify that $l = n = 1, m = 2$ in Theorem 9.1, then we get decomposition (9.3).

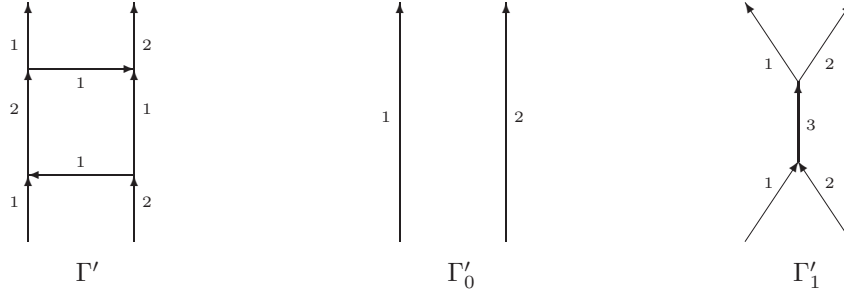


FIGURE 44.

To prove Theorem 9.1, we need the following special case of Proposition 7.20.

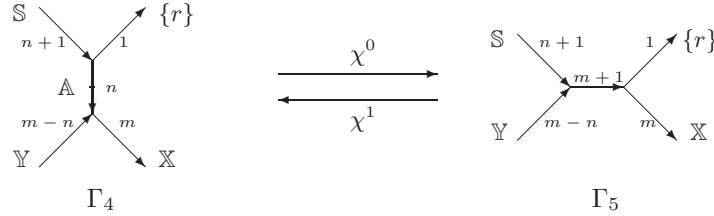


FIGURE 45.

Corollary 9.3. *Let Γ_4 and Γ_5 be the colored MOY graphs in Figure 45, where m, n are integers such that $0 \leq n \leq m \leq N$. Then there exist homogeneous morphisms*

$$\begin{aligned}\chi^0 &: C(\Gamma_4) \rightarrow C(\Gamma_5), \\ \chi^1 &: C(\Gamma_5) \rightarrow C(\Gamma_4),\end{aligned}$$

both of quantum degree $m - n$ and \mathbb{Z}_2 -degree 0 such that

$$\begin{aligned}\chi^1 \circ \chi^0 &\simeq \left(\sum_{k=0}^{m-n} (-r)^{m-n-k} Y_k \right) \cdot \text{id}_{C(\Gamma_4)}, \\ \chi^0 \circ \chi^1 &\simeq \left(\sum_{k=0}^{m-n} (-r)^{m-n-k} Y_k \right) \cdot \text{id}_{C(\Gamma_5)},\end{aligned}$$

where Y_k is the k -th elementary symmetric polynomial in \mathbb{Y} .

9.1. Relating Γ and Γ_0 . Consider the diagram in Figure 46, in which the ϕ and $\bar{\phi}$ are the morphisms associated to the apparent edge splitting and merging, h_0 and h_1 are the homotopy equivalence induced by the apparent bouquet moves, χ^0 and χ^1 are the morphisms from applying Corollary 9.3 to the left half of Γ . All these morphisms are $\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})$ -linear. Moreover, h_0 , h_1 , χ^0 and χ^1 are also $\text{Sym}(\mathbb{A}|\mathbb{Y})$ -linear. By Corollary 9.3, we know

$$(9.4) \quad \chi^1 \circ \chi^0 = \left(\sum_{k=0}^n (-r)^k A_{n-k} \right) \cdot \text{id}_{C(\Gamma_{10})}.$$

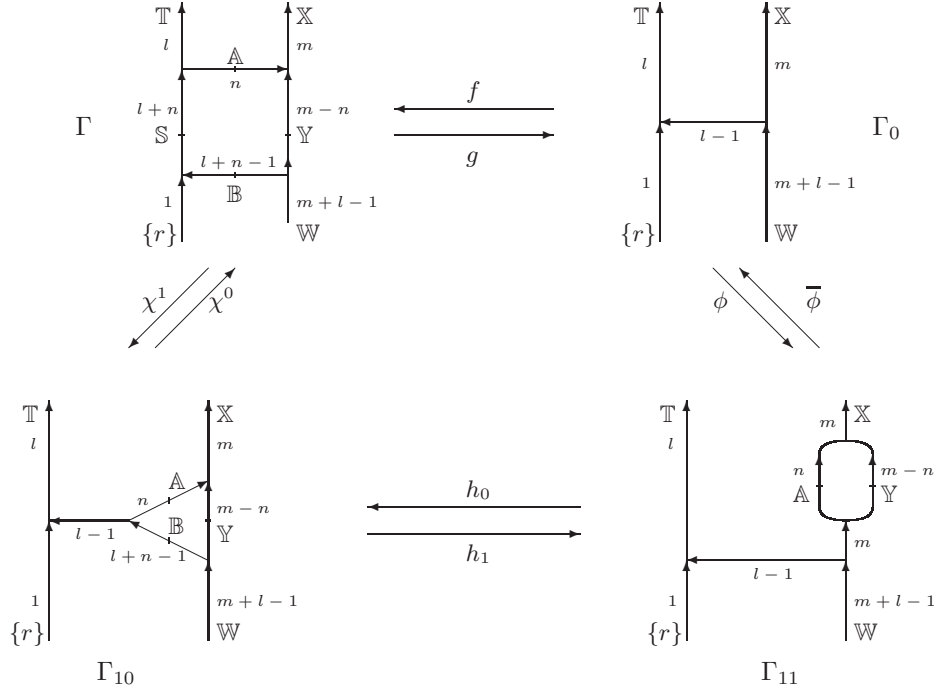


FIGURE 46.

Definition 9.4. Define $f : C(\Gamma_0) \rightarrow C(\Gamma)$ by $f = \chi^0 \circ h_0 \circ \phi$, and $g : C(\Gamma) \rightarrow C(\Gamma_0)$ by $g = \bar{\phi} \circ h_1 \circ \chi^1$.

Note that f and g are both homogeneous morphisms with quantum degree $-n(m-n-1)$ and \mathbb{Z}_2 -degree 0.

Let $\Lambda = \Lambda_{n,m-n-1} = \{\lambda \mid l(\lambda) \leq n, \lambda_1 \leq m-n-1\}$. For $\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \Lambda$, define $\lambda^c = (\lambda_1^c \geq \dots \geq \lambda_n^c) \in \Lambda$ by $\lambda_j^c = m-n-1 - \lambda_{n+1-j}$, $j = 1, \dots, n$.

Definition 9.5. For $\lambda \in \Lambda$, define $f_\lambda : C(\Gamma_0) \rightarrow C(\Gamma)$ by $f_\lambda = \mathbf{m}(S_\lambda(\mathbb{A})) \circ f$, where $S_\lambda(\mathbb{A})$ is the Schur polynomial in \mathbb{A} associated to λ , and $\mathbf{m}(S_\lambda(\mathbb{A}))$ is the morphism given by multiplication of $S_\lambda(\mathbb{A})$. f_λ is a homogeneous morphism with quantum degree $2|\lambda| - n(m-n-1)$ and \mathbb{Z}_2 -degree 0.

Also, define $g_\lambda : C(\Gamma) \rightarrow C(\Gamma_0)$ by $g_\lambda = g \circ \mathbf{m}(S_{\lambda^c}(-\mathbb{Y}))$, where $S_{\lambda^c}(-\mathbb{Y})$ is the Schur polynomial in $-\mathbb{Y}$ associated to λ^c . g_λ is a homogeneous morphism with quantum degree $n(m-n-1) - 2|\lambda|$ and \mathbb{Z}_2 -degree 0.

Lemma 9.6. Let \mathbb{A} be an alphabet with n indeterminants. Denote by A_k the k -th elementary symmetric polynomial in \mathbb{A} . For any $k = 1, \dots, n$ and any partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$, there is an expansion

$$A_k \cdot S_\lambda(\mathbb{A}) = \sum_{l(\mu) \leq n} c_\mu \cdot S_\mu(\mathbb{A}),$$

where $c_\mu \in \mathbb{Z}_{\geq 0}$. If $c_\mu \neq 0$, then $|\mu| - |\lambda| = k$ and $\lambda_j \leq \mu_j \leq \lambda_j + 1 \ \forall j = 1, \dots, n$. In particular,

$$A_n \cdot S_\lambda(\mathbb{A}) = S_{(\lambda_1+1 \geq \lambda_2+1 \geq \dots \geq \lambda_n+1)}(\mathbb{A}).$$

Proof. Note that $A_k = S_{\lambda_{k,1}}(\mathbb{A}) = S_{\underbrace{(1 \geq \dots \geq 1)}_{k \text{ parts}}}(\mathbb{A})$. This lemma is a special case of the Littlewood-Richardson rule (see e.g. [11, Appendix A]). \square

Lemma 9.7. *For $\lambda, \mu \in \Lambda$,*

$$g_\mu \circ f_\lambda \approx \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda < \mu. \end{cases}$$

Proof. For $\lambda, \mu \in \Lambda$, by (9.4), we have

$$\begin{aligned} g_\mu \circ f_\lambda &= g \circ \mathfrak{m}(S_{\mu^c}(-\mathbb{Y})) \circ \mathfrak{m}(S_\lambda(\mathbb{A})) \circ f \\ &= \bar{\phi} \circ h_1 \circ \chi^1 \circ \mathfrak{m}(S_{\mu^c}(-\mathbb{Y}) \cdot S_\lambda(\mathbb{A})) \circ \chi^0 \circ h_0 \circ \phi \\ &= \bar{\phi} \circ h_1 \circ \chi^1 \circ \chi^0 \circ h_0 \circ \mathfrak{m}(S_{\mu^c}(-\mathbb{Y}) \cdot S_\lambda(\mathbb{A})) \circ \phi \\ &\simeq \bar{\phi} \circ \mathfrak{m}\left(\left(\sum_{k=0}^n (-r)^k A_{n-k}\right) \cdot S_\lambda(\mathbb{A}) \cdot S_{\mu^c}(-\mathbb{Y})\right) \circ \phi. \end{aligned}$$

Write $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ and $\tilde{\lambda} = (\lambda_1 + 1 \geq \dots \geq \lambda_n + 1)$. By Lemma 9.6, we know that

$$\left(\sum_{k=0}^n (-r)^k A_{n-k}\right) \cdot S_\lambda(\mathbb{A}) = S_{\tilde{\lambda}}(\mathbb{A}) + \sum_{\lambda \leq \nu < \tilde{\lambda}} c_\nu(r) \cdot S_\nu(\mathbb{A}),$$

where $c_\nu(r) \in \mathbb{Z}[r]$. So

$$g_\mu \circ f_\lambda \simeq \bar{\phi} \circ \mathfrak{m}(S_{\tilde{\lambda}}(\mathbb{A}) \cdot S_{\mu^c}(-\mathbb{Y})) \circ \phi + \sum_{\lambda \leq \nu < \tilde{\lambda}} c_\nu(r) \cdot \bar{\phi} \circ \mathfrak{m}(S_\nu(\mathbb{A}) \cdot S_{\mu^c}(-\mathbb{Y})) \circ \phi.$$

Now the lemma follows from Lemma 7.11. \square

Lemma 9.8. *There exist homogeneous morphisms $F : C(\Gamma_0)\left\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\right\} \rightarrow C(\Gamma)$ and $G : C(\Gamma) \rightarrow C(\Gamma_0)\left\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\right\}$ preserving both gradings such that $G \circ F \simeq \text{id}_{C(\Gamma_0)\left\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\right\}}$.*

Proof. Note that

$$C(\Gamma_0)\left\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\right\} = \bigoplus_{\lambda \in \Lambda} C(\Gamma_0)\{q^{2|\lambda| - n(m-n-1)}\}.$$

We view f_λ as a homogeneous morphism $f_\lambda : C(\Gamma_0)\{q^{2|\lambda| - n(m-n-1)}\} \rightarrow C(\Gamma)$ preserving both gradings and g_λ as a homogeneous morphism $g_\lambda : C(\Gamma) \rightarrow C(\Gamma_0)\{q^{2|\lambda| - n(m-n-1)}\}$ preserving both gradings. Also, by choosing appropriate constants, we make

$$(9.5) \quad g_\mu \circ f_\lambda \simeq \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda < \mu. \end{cases}$$

Define $H_{\mu\lambda} : C(\Gamma_0)\{q^{2|\lambda| - n(m-n-1)}\} \rightarrow C(\Gamma_0)\{q^{2|\mu| - n(m-n-1)}\}$ by

$$H_{\mu\lambda} = \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda < \mu, \\ \sum_{k \geq 1} \sum_{\mu < \nu_1 < \dots < \nu_{k-1} < \lambda} (-1)^k (g_\mu \circ f_{\nu_1}) \circ (g_{\nu_1} \circ f_{\nu_2}) \circ \dots \circ (g_{\nu_{k-2}} \circ f_{\nu_{k-1}}) \circ (g_{\nu_{k-1}} \circ f_\lambda) & \text{if } \lambda > \mu. \end{cases}$$

Then define $\tilde{g}_\mu : C(\Gamma) \rightarrow C(\Gamma_0)\{q^{2|\mu| - n(m-n-1)}\}$ by

$$\tilde{g}_\mu = \sum_{\nu \geq \mu} H_{\mu\nu} \circ g_\nu.$$

Note that \tilde{g}_μ is a homogeneous morphism preserving both gradings.

Next consider $\tilde{g}_\mu \circ f_\lambda$.

(i) Suppose $\lambda < \mu$. Then, by (9.5),

$$\tilde{g}_\mu \circ f_\lambda = \sum_{\nu \geq \mu} H_{\mu\nu} \circ g_\nu \circ f_\lambda \simeq 0.$$

(ii) Suppose $\lambda = \mu$. Then, by (9.5),

$$\tilde{g}_\mu \circ f_\lambda = \sum_{\nu \geq \mu} H_{\mu\nu} \circ g_\nu \circ f_\mu \simeq H_{\mu\mu} \circ g_\mu \circ f_\mu \simeq \text{id}_{C(\Gamma_0)}.$$

(iii) Suppose $\lambda > \mu$. Then

$$\begin{aligned} \tilde{g}_\mu \circ f_\lambda &= \sum_{\nu \geq \mu} H_{\mu\nu} \circ g_\nu \circ f_\lambda \\ &\simeq H_{\mu\lambda} \circ g_\lambda \circ f_\lambda + H_{\mu\mu} \circ g_\mu \circ f_\lambda + \sum_{\mu < \nu < \lambda} H_{\mu\nu} \circ g_\nu \circ f_\lambda \\ &\simeq H_{\mu\lambda} + g_\mu \circ f_\lambda \\ &\quad + \sum_{k \geq 1} \sum_{\mu < \nu_1 < \dots < \nu_{k-1} < \nu < \lambda} (-1)^k (g_\mu \circ f_{\nu_1}) \circ (g_{\nu_1} \circ f_{\nu_2}) \circ \dots \circ (g_{\nu_{k-1}} \circ f_\nu) \circ (g_\nu \circ f_\lambda) \\ &= H_{\mu\lambda} - H_{\mu\lambda} = 0 \end{aligned}$$

Now define

$$F : C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} (= \bigoplus_{\lambda \in \Lambda} C(\Gamma_0) \{q^{2|\lambda| - n(m-n-1)}\}) \rightarrow C(\Gamma)$$

by

$$F = \sum_{\lambda \in \Lambda} f_\lambda,$$

and

$$G : C(\Gamma) \rightarrow C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} (= \bigoplus_{\lambda \in \Lambda} C(\Gamma_0) \{q^{2|\lambda| - n(m-n-1)}\})$$

by

$$G = \sum_{\lambda \in \Lambda} \tilde{g}_\lambda.$$

Then F and G are homogeneous morphism preserving both gradings, and

$$G \circ F \simeq \text{id}_{C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\}}.$$

□

9.2. Relating Γ and Γ_1 . Consider the diagram in Figure 47, in which the ϕ and $\bar{\phi}$ are the morphisms associated to the apparent edge splitting and merging, h_0 and h_1 are the homotopy equivalence induced by the apparent bouquet moves, χ^0 and χ^1 are the morphisms from applying Corollary 9.3 to the lower half of Γ . All these morphisms are $\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})$ -linear. Moreover, h_0 , h_1 , χ^0 and χ^1 are also $\text{Sym}(\mathbb{A}|\mathbb{Y})$ -linear. By Corollary 9.3, we know

$$(9.6) \quad \chi^0 \circ \chi^1 = \left(\sum_{k=0}^{m-n} (-r)^k Y_{m-n-k} \right) \cdot \text{id}_{C(\Gamma_{12})}.$$

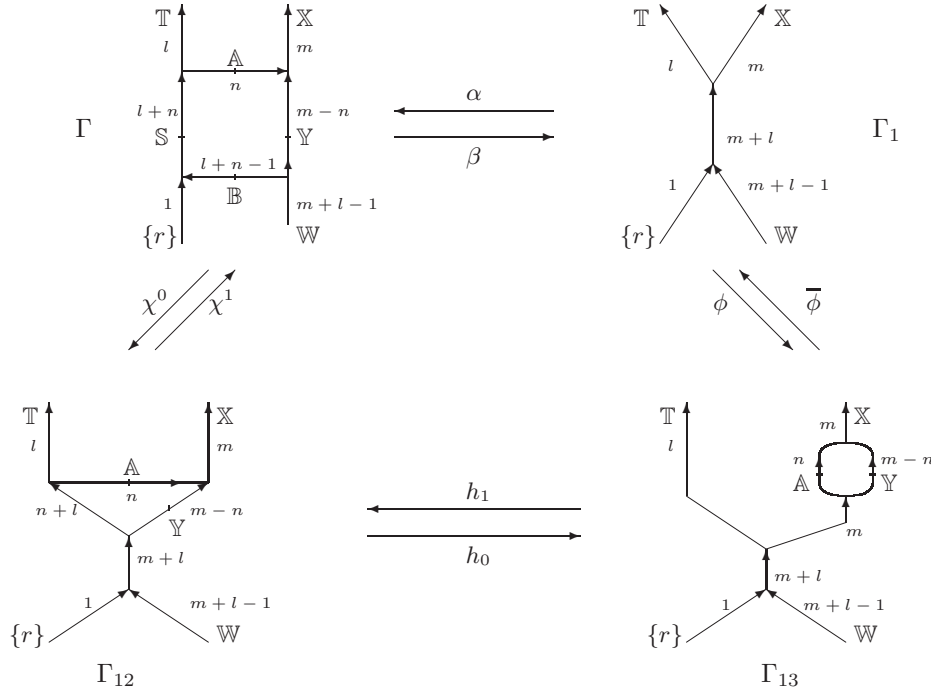


FIGURE 47.

Definition 9.9. Define $\alpha : C(\Gamma_1) \rightarrow C(\Gamma)$ by $\alpha = \chi^1 \circ h_1 \circ \phi$, and $\beta : C(\Gamma) \rightarrow C(\Gamma_1)$ by $\beta = \bar{\phi} \circ h_0 \circ \chi^0$.

Note that α and β are both homogeneous morphisms with quantum degree $-(n-1)(m-n)$ and \mathbb{Z}_2 -degree 0.

Let $\Lambda' = \Lambda_{m-n, n-1} = \{\lambda \mid l(\lambda) \leq m-n, \lambda_1 \leq n-1\}$. For $\lambda = (\lambda_1 \geq \dots \geq \lambda_{m-n}) \in \Lambda'$, define $\lambda^* = (\lambda_1^* \geq \dots \geq \lambda_{m-n}^*) \in \Lambda'$ by $\lambda_j^* = n-1 - \lambda_{m-n+1-j}$, $j = 1, \dots, m-n$.

Definition 9.10. For $\lambda \in \Lambda'$, define $\alpha_\lambda : C(\Gamma_1) \rightarrow C(\Gamma)$ by $\alpha_\lambda = \mathbf{m}(S_\lambda(\mathbb{Y})) \circ \alpha$, where $S_\lambda(\mathbb{Y})$ is the Schur polynomial in \mathbb{Y} associated to λ . α_λ is a homogeneous morphism with quantum degree $2|\lambda| - (n-1)(m-n)$ and \mathbb{Z}_2 -degree 0.

Also, define $\beta_\lambda : C(\Gamma) \rightarrow C(\Gamma_1)$ by $\beta_\lambda = \beta \circ \mathbf{m}(S_{\lambda^*}(-\mathbb{A}))$, where $S_{\lambda^*}(-\mathbb{A})$ is the Schur polynomial in $-\mathbb{A}$ associated to λ^* . β_λ is a homogeneous morphism with quantum degree $(n-1)(m-n) - 2|\lambda|$ and \mathbb{Z}_2 -degree 0.

Lemma 9.11. For $\lambda, \mu \in \Lambda'$,

$$\beta_\mu \circ \alpha_\lambda \approx \begin{cases} \text{id}_{C(\Gamma_1)} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda < \mu. \end{cases}$$

Proof. For $\lambda, \mu \in \Lambda'$, by (9.6), we have

$$\begin{aligned} \beta_\mu \circ \alpha_\lambda &= \beta \circ \mathbf{m}(S_{\mu^*}(-\mathbb{A})) \circ \mathbf{m}(S_\lambda(\mathbb{Y})) \circ \alpha \\ &= \bar{\phi} \circ h_0 \circ \chi^0 \circ \mathbf{m}(S_{\mu^*}(-\mathbb{A}) \cdot S_\lambda(\mathbb{Y})) \circ \chi^1 \circ h_1 \circ \phi \\ &= \bar{\phi} \circ h_0 \circ \chi^0 \circ \chi^1 \circ h_1 \circ \mathbf{m}(S_{\mu^*}(-\mathbb{A}) \cdot S_\lambda(\mathbb{Y})) \circ \phi \\ &\simeq \bar{\phi} \circ \mathbf{m}\left(\sum_{k=0}^{m-n} (-r)^k Y_{m-n-k}\right) \cdot S_\lambda(\mathbb{Y}) \cdot S_{\mu^*}(-\mathbb{A}) \circ \phi. \end{aligned}$$

Write $\lambda = (\lambda_1 \geq \dots \geq \lambda_{m-n})$ and $\tilde{\lambda} = (\lambda_1 + 1 \geq \dots \geq \lambda_{m-n} + 1)$. By Lemma 9.6, we know that

$$\left(\sum_{k=0}^{m-n} (-r)^k Y_{m-n-k}\right) \cdot S_\lambda(\mathbb{Y}) = S_{\tilde{\lambda}}(\mathbb{Y}) + \sum_{\lambda \leq \nu < \tilde{\lambda}} c_\nu(r) \cdot S_\nu(\mathbb{Y}),$$

where $c_\nu(r) \in \mathbb{Z}[r]$. So

$$\beta_\mu \circ \alpha_\lambda \simeq \bar{\phi} \circ \mathbf{m}(S_{\tilde{\lambda}}(\mathbb{Y}) \cdot S_{\mu^*}(-\mathbb{A})) \circ \phi + \sum_{\lambda \leq \nu < \tilde{\lambda}} c_\nu(r) \cdot \bar{\phi} \circ \mathbf{m}(S_\nu(\mathbb{Y}) \cdot S_{\mu^*}(-\mathbb{A})) \circ \phi.$$

Now the lemma follows from Lemma 7.11. \square

Lemma 9.12. *There exist homogeneous morphisms $\vec{\alpha} : C(\Gamma_1)\{[m-1]_{n-1}\} \rightarrow C(\Gamma)$ and $\vec{\beta} : C(\Gamma) \rightarrow C(\Gamma_1)\{[m-1]_{n-1}\}$ preserving both gradings such that $\vec{\beta} \circ \vec{\alpha} \simeq \text{id}_{C(\Gamma_1)\{[m-1]_{n-1}\}}$.*

Proof. Note that

$$C(\Gamma_1)\left\{\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right\} = \bigoplus_{\lambda \in \Lambda'} C(\Gamma_1)\{q^{2|\lambda|-(n-1)(m-n)}\}.$$

We view α_λ as a homogeneous morphism $\alpha_\lambda : C(\Gamma_1)\{q^{2|\lambda|-(n-1)(m-n)}\} \rightarrow C(\Gamma)$ preserving both gradings and β_λ as a homogeneous morphism $\beta_\lambda : C(\Gamma) \rightarrow C(\Gamma_1)\{q^{2|\lambda|-(n-1)(m-n)}\}$ preserving both gradings. Also, by choosing appropriate constants, we make

$$(9.7) \quad \beta_\mu \circ \alpha_\lambda \simeq \begin{cases} \text{id}_{C(\Gamma_1)} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda < \mu. \end{cases}$$

Define $\tau_{\mu\lambda} : C(\Gamma_1)\{q^{2|\lambda|-(n-1)(m-n)}\} \rightarrow C(\Gamma_1)\{q^{2|\mu|-(n-1)(m-n)}\}$ by

$$\tau_{\mu\lambda} = \begin{cases} \text{id}_{C(\Gamma_1)} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda < \mu, \\ \sum_{k \geq 1} \sum_{\mu < \nu_1 < \dots < \nu_{k-1} < \lambda} (-1)^k (\beta_\mu \circ \alpha_{\nu_1}) \circ (\beta_{\nu_1} \circ \alpha_{\nu_2}) \circ \dots \circ (\beta_{\nu_{k-2}} \circ \alpha_{\nu_{k-1}}) \circ (\beta_{\nu_{k-1}} \circ \alpha_\lambda) & \text{if } \lambda > \mu. \end{cases}$$

Then define $\tilde{\beta}_\mu : C(\Gamma) \rightarrow C(\Gamma_0)\{q^{2|\mu|-(n-1)(m-n)}\}$ by

$$\tilde{\beta}_\mu = \sum_{\nu \geq \mu} \tau_{\mu\nu} \circ \beta_\nu.$$

Note that $\tilde{\beta}_\mu$ is a homogeneous morphism preserving both gradings.

Next consider $\tilde{\beta}_\mu \circ \alpha_\lambda$.

(i) Suppose $\lambda < \mu$. Then, by (9.7),

$$\tilde{\beta}_\mu \circ \alpha_\lambda = \sum_{\nu \geq \mu} \tau_{\mu\nu} \circ \beta_\nu \circ \alpha_\lambda \simeq 0.$$

(ii) Suppose $\lambda = \mu$. Then, by (9.7),

$$\tilde{\beta}_\mu \circ \alpha_\lambda = \sum_{\nu \geq \mu} \tau_{\mu\nu} \circ \beta_\nu \circ \alpha_\mu \simeq \tau_{\mu\mu} \circ \beta_\mu \circ \alpha_\mu \simeq \text{id}_{C(\Gamma_1)}.$$

(iii) Suppose $\lambda > \mu$. Then

$$\begin{aligned} \tilde{\beta}_\mu \circ \alpha_\lambda &= \sum_{\nu \geq \mu} \tau_{\mu\nu} \circ \beta_\nu \circ \alpha_\lambda \\ &\simeq \tau_{\mu\lambda} \circ \beta_\lambda \circ \alpha_\lambda + \tau_{\mu\mu} \circ \beta_\mu \circ \alpha_\lambda + \sum_{\mu < \nu < \lambda} \tau_{\mu\nu} \circ \beta_\nu \circ \alpha_\lambda \\ &\simeq \tau_{\mu\lambda} + \beta_\mu \circ \alpha_\lambda \\ &\quad + \sum_{k \geq 1} \sum_{\mu < \nu_1 < \dots < \nu_{k-1} < \nu < \lambda} (-1)^k (\beta_\mu \circ \alpha_{\nu_1}) \circ (\beta_{\nu_1} \circ \alpha_{\nu_2}) \circ \dots \circ (\beta_{\nu_{k-1}} \circ \alpha_\nu) \circ (\beta_\nu \circ \alpha_\lambda) \\ &= \tau_{\mu\lambda} - \tau_{\mu\lambda} = 0 \end{aligned}$$

Now define

$$\vec{\alpha} : C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\} = \bigoplus_{\lambda \in \Lambda'} C(\Gamma_1) \{q^{2|\lambda| - (n-1)(m-n)}\} \rightarrow C(\Gamma)$$

by

$$\vec{\alpha} = \sum_{\lambda \in \Lambda'} \alpha_\lambda,$$

and

$$\vec{\beta} : C(\Gamma) \rightarrow C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\} = \bigoplus_{\lambda \in \Lambda'} C(\Gamma_1) \{q^{2|\lambda| - (n-1)(m-n)}\}$$

by

$$\vec{\beta} = \sum_{\lambda \in \Lambda'} \tilde{\beta}_\lambda.$$

Then $\vec{\alpha}$ and $\vec{\beta}$ are homogeneous morphism preserving both gradings, and

$$\vec{\beta} \circ \vec{\alpha} \simeq \text{id}_{C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\}}.$$

□

9.3. Homotopic nilpotency of $\vec{\beta} \circ F \circ G \circ \vec{\alpha}$ and $G \circ \vec{\alpha} \circ \vec{\beta} \circ F$.

Lemma 9.13. *Let Γ_0 and Γ_1 be as in Figure 43. Then*

$$\begin{aligned} &\text{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1)) \cong \text{Hom}_{HMF}(C(\Gamma_1), C(\Gamma_0)) \\ &\cong C(\emptyset) \left\{ \begin{bmatrix} l+m-1 \\ m \end{bmatrix} \cdot \begin{bmatrix} l+m \\ 1 \end{bmatrix} \cdot \begin{bmatrix} N \\ l+m \end{bmatrix} \cdot q^{(l+m)(N+1-l-m)+ml-1} \right\}, \end{aligned}$$

where $C(\emptyset)$ is $\mathbb{C} \rightarrow 0 \rightarrow \mathbb{C}$. In particular, the lowest non-vanishing quantum grading of these spaces is m .

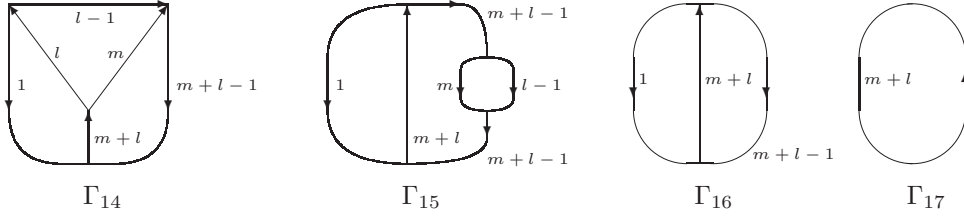


FIGURE 48.

Proof. Mark Γ_0 and Γ_1 as in Figure 43. Then

$$C(\Gamma_0) = \begin{pmatrix} * & T_1 - r - D_1 \\ \cdots & \cdots \\ * & T_k - rD_{k-1} - D_k \\ \cdots & \cdots \\ * & T_l - rD_{l-1} \\ * & D_1 + X_1 - W_1 \\ \cdots & \cdots \\ * & \sum_{j=0}^k D_j X_{k-j} - W_k \\ \cdots & \cdots \\ * & D_{l-1} X_m - W_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{D}|\mathbb{T}|\{r\})} \{q^{-(l-1)m}\},$$

where X_k the the k -th elementary symmetric polynomial in \mathbb{X} and so on. By Proposition 2.19, we exclude D_1, \dots, D_{l-1} from this matrix factorization using the right entries of the first $l-1$ rows. We get the relation

$$D_k = \begin{cases} \sum_{j=0}^k (-r)^j T_{k-j} & \text{if } 0 \leq k \leq l-1, \\ 0 & \text{if } k < 0 \text{ or } k > l-1, \end{cases}$$

and

$$C(\Gamma_0) \simeq \begin{pmatrix} * & \sum_{j=0}^l (-r)^j T_{l-j} \\ * & T_1 - r + X_1 - W_1 \\ \cdots & \cdots \\ * & \sum_{j=0}^{l-1} \sum_{i=0}^j (-r)^i T_{j-i} X_{k-j} - W_k \\ \cdots & \cdots \\ * & \sum_{i=0}^{l-1} (-r)^i T_{l-1-i} X_m - W_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})} \{q^{-(l-1)m}\}.$$

So

$$C(\Gamma_0)_\bullet \simeq \begin{pmatrix} * & -\sum_{j=0}^l (-r)^j T_{l-j} \\ * & -(T_1 - r + X_1 - W_1) \\ \cdots & \cdots \\ * & -(\sum_{j=0}^{l-1} \sum_{i=0}^j (-r)^i T_{j-i} X_{k-j} - W_k) \\ \cdots & \cdots \\ * & -(\sum_{i=0}^{l-1} (-r)^i T_{l-1-i} X_m - W_{m+l-1}) \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})} \{q^{(l+m)(N+1-l-m)+(l-1)m}\} \langle l+m \rangle.$$

Let $\bar{\Gamma}_0$ be Γ_0 with the orientation of every edge reversed. Similar to above, we have

$$C(\bar{\Gamma}_0) \simeq \left(\begin{array}{cc} * & -\sum_{j=0}^l (-r)^j T_{l-j} \\ * & -(T_1 - r + X_1 - W_1) \\ \cdots & \cdots \\ * & -(\sum_{j=0}^{l-1} \sum_{i=0}^j (-r)^i T_{j-i} X_{k-j} - W_k) \\ \cdots & \cdots \\ * & -(\sum_{i=0}^{l-1} (-r)^i T_{l-1-i} X_m - W_{m+l-1}) \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})} \{q^{-l+1}\}.$$

Thus, $C(\Gamma_0)_\bullet \simeq C(\bar{\Gamma}_0)\{q^{(l+m)(N+1-l-m)+lm-1}\} \langle l+m \rangle$ and, therefore,

$$\text{Hom}(C(\Gamma_0), C(\Gamma_1)) \cong C(\Gamma_1) \otimes C(\Gamma_0)_\bullet \simeq C(\Gamma_1) \otimes C(\bar{\Gamma}_0)\{q^{(l+m)(N+1-l-m)+lm-1}\} \langle l+m \rangle.$$

Let $\Gamma_{14}, \dots, \Gamma_{17}$ be the MOY graphs in Figure 48. Then

$$\begin{aligned} C(\bar{\Gamma}_0) \otimes C(\Gamma_1) &\simeq C(\Gamma_{14}) \simeq C(\Gamma_{15}) \simeq C(\Gamma_{16})\left\{\begin{bmatrix} m+l-1 \\ m \end{bmatrix}\right\} \\ &\simeq C(\Gamma_{17})\left\{\begin{bmatrix} m+l \\ 1 \end{bmatrix} \cdot \begin{bmatrix} m+l-1 \\ m \end{bmatrix}\right\} \\ &\simeq C(\emptyset)\left\{\begin{bmatrix} N \\ m+l \end{bmatrix} \cdot \begin{bmatrix} m+l \\ 1 \end{bmatrix} \cdot \begin{bmatrix} m+l-1 \\ m \end{bmatrix}\right\}. \end{aligned}$$

So

$$\text{Hom}_{HMF}(C(\Gamma_0), C(\Gamma_1)) \cong C(\emptyset)\left\{\begin{bmatrix} l+m-1 \\ m \end{bmatrix} \cdot \begin{bmatrix} l+m \\ 1 \end{bmatrix} \cdot \begin{bmatrix} N \\ l+m \end{bmatrix} \cdot q^{(l+m)(N+1-l-m)+ml-1}\right\}.$$

The computation of $\text{Hom}_{HMF}(C(\Gamma_1), C(\Gamma_0))$ is very similar. Using the fact that

$$C(\Gamma_1) \simeq \left(\begin{array}{cc} * & T_1 + X_1 - r - W_1 \\ \cdots & \cdots \\ * & \sum_{j=0}^k T_j X_{k-j} - r W_{k-1} - W_k \\ \cdots & \cdots \\ * & T_l X_m - r W_{m+l-1} \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})} \{q^{-lm}\},$$

one gets $C(\Gamma_1)_\bullet \simeq C(\bar{\Gamma}_1)\{q^{(l+m)(N+1-l-m)+lm-1}\} \langle l+m \rangle$, where $\bar{\Gamma}_1$ is Γ_1 with the orientation on each edges reversed. So

$$\begin{aligned} \text{Hom}(C(\Gamma_1), C(\Gamma_0)) &\cong C(\Gamma_0) \otimes C(\Gamma_1)_\bullet \\ &\simeq C(\Gamma_0) \otimes C(\bar{\Gamma}_1)\{q^{(l+m)(N+1-l-m)+lm-1}\} \langle l+m \rangle \\ &\simeq C(\bar{\Gamma}_{14})\{q^{(l+m)(N+1-l-m)+lm-1}\} \langle l+m \rangle \\ &\simeq \cdots \simeq \\ &\simeq C(\emptyset)\left\{\begin{bmatrix} l+m-1 \\ m \end{bmatrix} \cdot \begin{bmatrix} l+m \\ 1 \end{bmatrix} \cdot \begin{bmatrix} N \\ l+m \end{bmatrix} \cdot q^{(l+m)(N+1-l-m)+ml-1}\right\}, \end{aligned}$$

where $\bar{\Gamma}_{14}$ is Γ_{14} with the orientation on each edges reversed. \square

Lemma 9.14. *For $\mu \in \Lambda$ and $\lambda \in \Lambda'$, let α_λ , $\tilde{\beta}_\lambda$, f_μ and \tilde{g}_μ be the morphisms defined in the two preceding subsections. We have*

- If $|\lambda| - |\mu| < n$, then $\tilde{g}_\mu \circ \alpha_\lambda \simeq 0$.
- If $|\mu| - |\lambda| < m - n$, then $\tilde{\beta}_\lambda \circ f_\mu \simeq 0$.

Proof. Note that $\tilde{g}_\mu \circ \alpha_\lambda : C(\Gamma_1) \rightarrow C(\Gamma_0)$ is a homogeneous morphism of quantum degree

$$2|\lambda| - (n-1)(m-n) - 2|\mu| + n(m-n-1) = 2(|\lambda| - |\mu| - n) + m,$$

and $\tilde{\beta}_\lambda \circ f_\mu : C(\Gamma_0) \rightarrow C(\Gamma_1)$ is a homogeneous morphism of quantum degree

$$-2|\lambda| + (n-1)(m-n) + 2|\mu| - n(m-n-1) = 2(|\mu| - |\lambda| - (m-n)) + m.$$

Then the lemma follows from Lemma 9.13. \square

Lemma 9.15. *Let $\vec{\alpha}$, $\vec{\beta}$, F and G be the morphisms defined in the two preceding subsections. Then $\vec{\beta} \circ F \circ G \circ \vec{\alpha}$ and $G \circ \vec{\alpha} \circ \vec{\beta} \circ F$ are both homotopically nilpotent.*

Proof. For $\lambda, \mu \in \Lambda'$, the (μ, λ) -component of $(\vec{\beta} \circ F \circ G \circ \vec{\alpha})^k$ is

$$\sum_{\lambda_1, \dots, \lambda_{k-1} \in \Lambda', \nu_1, \dots, \nu_k \in \Lambda} (\tilde{\beta}_\mu \circ f_{\nu_1} \circ \tilde{g}_{\nu_1} \circ \alpha_{\lambda_1}) \circ (\tilde{\beta}_{\lambda_1} \circ f_{\nu_2} \circ \tilde{g}_{\nu_2} \circ \alpha_{\lambda_2}) \circ \dots \circ (\tilde{\beta}_{\lambda_{k-1}} \circ f_{\nu_k} \circ \tilde{g}_{\nu_k} \circ \alpha_\lambda).$$

By Lemma 9.14, for the term corresponding to $\lambda_1, \dots, \lambda_{k-1} \in \Lambda', \nu_1, \dots, \nu_k \in \Lambda$ to be homotopically non-vanishing, we must have

$$\begin{aligned} |\lambda| - |\nu_k| &\geq n, \\ |\nu_1| - |\mu| &\geq m - n, \\ |\lambda_j| - |\nu_j| &\geq n, \text{ for } j = 1, \dots, k-1, \\ |\nu_{j+1}| - |\lambda_j| &\geq m - n, \text{ for } j = 1, \dots, k-1. \end{aligned}$$

Adding all these inequalities together, we get $|\lambda| - |\mu| \geq km$. Note that $|\lambda| - |\mu| \leq (n-1)(m-n)$. This implies that $(\vec{\beta} \circ F \circ G \circ \vec{\alpha})^k \simeq 0$ if $km > (n-1)(m-n)$. Thus, $\vec{\beta} \circ F \circ G \circ \vec{\alpha}$ is homotopically nilpotent. Since

$$(G \circ \vec{\alpha} \circ \vec{\beta} \circ F)^{k+1} = G \circ \vec{\alpha} \circ (\vec{\beta} \circ F \circ G \circ \vec{\alpha})^k \circ \vec{\beta} \circ F,$$

$G \circ \vec{\alpha} \circ \vec{\beta} \circ F$ is also homotopically nilpotent. \square

9.4. Graded dimensions of $C(\Gamma)$, $C(\Gamma_0)$ and $C(\Gamma_1)$.

Lemma 9.16. *Let Γ , Γ_0 and Γ_1 be the MOY graphs in Figure 43, where l, m, n are integers satisfying $0 \leq n \leq m \leq N$ and $0 \leq l, m+l-1 \leq N$. Then*

$$\begin{aligned} \text{gdim} C(\Gamma_0) &= q^{-lm+m} (1 + \tau q^{2l-N-1}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}), \\ \text{gdim} C(\Gamma_1) &= \begin{cases} q^{-lm} \prod_{j=1}^{m+l} (1 + \tau q^{2j-N-1}) & \text{if } l+m \leq N, \\ 0 & \text{if } l+m = N+1, \end{cases} \\ \text{gdim} C(\Gamma) &= \begin{cases} q^{-lm+m-n} \begin{bmatrix} m \\ n \end{bmatrix} (1 + \tau q^{2n+2l-N-1}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}) & \text{if } l+m \leq N, \\ q^{-lm+m} \begin{bmatrix} m-1 \\ n \end{bmatrix} (1 + \tau q^{N+1-2m}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}) & \text{if } l+m = N+1. \end{cases} \end{aligned}$$

In particular,

$$(9.8) \quad \text{gdim} C(\Gamma) = \begin{bmatrix} m-1 \\ n \end{bmatrix} \cdot \text{gdim} C(\Gamma_0) + \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \cdot \text{gdim} C(\Gamma_1).$$

Proof. We mark Γ , Γ_0 and Γ_1 as in Figure 43. Then $C(\Gamma)$, $C(\Gamma_0)$ and $C(\Gamma_1)$ are matrix factorizations over $\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})$. The corresponding maximal ideal is

$$\mathfrak{I} = (X_1, \dots, X_m, W_1, \dots, W_{l+m-1}, T_1, \dots, T_l, r),$$

where X_j is the j -th elementary symmetric polynomial in \mathbb{X} and so on.

We compute $\text{gdim}C(\Gamma_0)$ first.

From the proof of Lemma 9.13, we know that

$$C(\Gamma_0) \simeq \left(\begin{array}{cc} * & \sum_{j=0}^l (-r)^j T_{l-j} \\ * & T_1 - r + X_1 - W_1 \\ \cdots & \cdots \\ * & \sum_{j=0}^k \sum_{i=0}^j (-r)^i T_{j-i} X_{k-j} - W_k \\ \cdots & \cdots \\ * & \sum_{i=0}^{l-1} (-r)^i T_{l-1-i} X_m - W_{m+l-1} \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})} \{q^{-(l-1)m}\}.$$

So

$$C(\Gamma_0)/\mathfrak{I} \cdot C(\Gamma_0) \simeq \left(\begin{array}{cc} 0 & 0_l \\ 0 & 0_1 \\ \cdots & \cdots \\ 0 & 0_{m+l-1} \end{array} \right)_{\mathbb{C}} \{q^{-(l-1)m}\},$$

where 0_j means “a 0 of degree $2j$ ”. then it follows easily that

$$\text{gdim}C(\Gamma_0) = q^{-lm+m} (1 + \tau q^{2l-N-1}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}).$$

Next we compute $\text{gdim}C(\Gamma_1)$.

If $l + m = N + 1$, then $C(\Gamma_1) \simeq 0$. So $\text{gdim}C(\Gamma_1) = 0$.

If $l + m \leq N$, then

$$C(\Gamma_1) \simeq \left(\begin{array}{cc} * & T_1 + X_1 - r - W_1 \\ \cdots & \cdots \\ * & \sum_{j=0}^k T_j X_{k-j} - r W_{k-1} - W_k \\ \cdots & \cdots \\ * & T_l X_m - r W_{m+l-1} \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{W}|\mathbb{T}|\{r\})} \{q^{-lm}\}$$

and, therefore,

$$C(\Gamma_1)/\mathfrak{I} \cdot C(\Gamma_1) \simeq \left(\begin{array}{cc} 0 & 0_1 \\ \cdots & \cdots \\ 0 & 0_{m+l} \end{array} \right)_{\mathbb{C}} \{q^{-lm}\}.$$

So

$$\text{gdim}C(\Gamma_1) = q^{-lm} \prod_{j=1}^{m+l} (1 + \tau q^{2j-N-1}).$$

Now we compute $C(\Gamma)$.

Let $\mathbb{D} = \mathbb{A} \cup \mathbb{T}$ and $\mathbb{E} = \{r\} \cup \mathbb{B}$. Denote by D_j and E_j the j -th elementary symmetric polynomials in \mathbb{D} and \mathbb{E} . Define

$$U_j = \frac{p_{l+n, N+1}(E_1, \dots, E_{j-1}, D_j, \dots, D_{l+n}) - p_{l+n, N+1}(E_1, \dots, E_j, D_{j+1}, \dots, D_{l+n})}{D_j - E_j},$$

$$\begin{aligned} V_j &= (-1)^{j-1} p_{N+1-j}(\mathbb{A} \cup \mathbb{Y}) + \sum_{k=1}^m (-1)^{k+j} j X_k h_{N+1-j-k}(\mathbb{A} \cup \mathbb{Y}) \\ &\quad + \sum_{k=1}^m \sum_{i=1}^m (-1)^{k+i} i X_k X_i \xi_{N+1-k-i, j}(\mathbb{X}, \mathbb{A} \cup \mathbb{Y}), \end{aligned}$$

$$\begin{aligned} \hat{V}_j &= (-1)^{j-1} p_{N+1-j}(\mathbb{B} \cup \mathbb{Y}) + \sum_{k=1}^{m+l-1} (-1)^{k+j} j W_k h_{N+1-j-k}(\mathbb{B} \cup \mathbb{Y}) \\ &\quad + \sum_{k=1}^{m+l-1} \sum_{i=1}^{m+l-1} (-1)^{k+i} i W_k W_i \xi_{N+1-k-i, j}(\mathbb{W}, \mathbb{B} \cup \mathbb{Y}), \end{aligned}$$

where $\xi_{k,j}$ is defined as in Lemma 7.38. Then, by Lemma 7.38, we have

$$C(\Gamma) \cong \begin{pmatrix} U_1 & D_1 - E_1 \\ \dots & \dots \\ U_{n+l} & D_{n+l} - E_{n+l} \\ V_1 & X_1 - A_1 - Y_1 \\ \dots & \dots \\ V_m & X_m - A_n Y_{m-n} \\ \hat{V}_1 & B_1 + Y_1 - W_1 \\ \dots & \dots \\ \hat{V}_{m+l-1} & B_{n+l-1} Y_{m-n} - W_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{W}|\mathbb{A}|\mathbb{B}|\mathbb{T}|\{r\})} \{q^{-ln-(l+n-1)(m-n)}\}.$$

Note that

$$\begin{aligned} V_j|_{X_1=\dots=X_m=0} &= (-1)^{j-1} p_{N+1-j}(\mathbb{A} \cup \mathbb{Y}), \\ \hat{V}_j|_{W_1=\dots=W_{m+l-1}=0} &= (-1)^{j-1} p_{N+1-j}(\mathbb{B} \cup \mathbb{Y}), \\ D_j|_{T_1=\dots=T_l=0} &= A_j, \\ E_j|_{r=0} &= B_j. \end{aligned}$$

So

$$C(\Gamma)/\mathfrak{I} \cdot C(\Gamma) \cong \begin{pmatrix} \tilde{U}_1 & A_1 - B_1 \\ \dots & \dots \\ \tilde{U}_n & A_n - B_n \\ \tilde{U}_{n+1} & -B_{n+1} \\ \dots & \dots \\ \tilde{U}_{n+l-1} & -B_{n+l-1} \\ \tilde{U}_{n+l} & 0 \\ p_N(\mathbb{A} \cup \mathbb{Y}) & -A_1 - Y_1 \\ \dots & \dots \\ (-1)^{m-1} p_{N+1-m}(\mathbb{A} \cup \mathbb{Y}) & -A_n Y_{m-n} \\ p_N(\mathbb{B} \cup \mathbb{Y}) & B_1 + Y_1 \\ \dots & \dots \\ (-1)^{m+l-2} p_{N+1-(m+l-1)}(\mathbb{B} \cup \mathbb{Y}) & B_{n+l-1} Y_{m-n} \end{pmatrix}_{\text{Sym}(\mathbb{Y}|\mathbb{A}|\mathbb{B})} \{q^{-ln-(l+n-1)(m-n)}\},$$

where

$$\tilde{U}_j = U_j|_{T_1=\dots=T_l=r=0}.$$

Next, we exclude B_1, \dots, B_{n+l-1} by applying Proposition 2.19 to the first $n+l-1$ rows of this matrix factorization. This gives the relation

$$B_j = A_j = \begin{cases} A_j & \text{if } 1 \leq j \leq n, \\ 0 & \text{if } n+1 \leq j \leq n+l-1, \end{cases}$$

and

$$C(\Gamma)/\mathcal{I} \cdot C(\Gamma) \simeq \begin{pmatrix} \tilde{U}_{n+l}|_{B_j=A_j} & 0 \\ p_N(\mathbb{A} \cup \mathbb{Y}) & -A_1 - Y_1 \\ \dots & \dots \\ (-1)^{m-1} p_{N+1-m}(\mathbb{A} \cup \mathbb{Y}) & -A_n Y_{m-n} \\ p_N(\mathbb{A} \cup \mathbb{Y}) & A_1 + Y_1 \\ \dots & \dots \\ (-1)^{m-1} p_{N+1-m}(\mathbb{A} \cup \mathbb{Y}) & A_m Y_{m-n} \\ (-1)^m p_{N+1-(m+1)}(\mathbb{A} \cup \mathbb{Y}) & 0_{m+1} \\ \dots & \dots \\ (-1)^{m+l-2} p_{N+1-(m+l-1)}(\mathbb{A} \cup \mathbb{Y}) & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{Y}|\mathbb{A})} \{q^{-ln-(l+n-1)(m-n)}\}.$$

By Corollary 2.16, we have

$$C(\Gamma)/\mathcal{I} \cdot C(\Gamma) \simeq \begin{pmatrix} \tilde{U}_{n+l}|_{B_j=A_j} & 0 \\ 0 & -A_1 - Y_1 \\ \dots & \dots \\ 0 & -A_n Y_{m-n} \\ p_N(\mathbb{A} \cup \mathbb{Y}) & 0_1 \\ \dots & \dots \\ (-1)^{m+l-2} p_{N+1-(m+l-1)}(\mathbb{A} \cup \mathbb{Y}) & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{Y}|\mathbb{A})} \{q^{-ln-(l+n-1)(m-n)}\}.$$

Since $m+l-1 \leq N$, $p_{N+1-(m+l-1)}(\mathbb{A} \cup \mathbb{Y}), \dots, p_N(\mathbb{A} \cup \mathbb{Y})$ belong to the ideal generated by $A_1 + Y_1, \dots, \sum_{j=0}^k A_j Y_{k-j}, \dots, A_n Y_{m-n}$. So, by Corollary 2.15, we have

$$C(\Gamma)/\mathcal{I} \cdot C(\Gamma) \simeq \begin{pmatrix} \tilde{U}_{n+l}|_{B_j=A_j} & 0 \\ 0 & -A_1 - Y_1 \\ \dots & \dots \\ 0 & -A_n Y_{m-n} \\ 0 & 0_1 \\ \dots & \dots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{Y}|\mathbb{A})} \{q^{-ln-(l+n-1)(m-n)}\}.$$

Note that, by Lemma 4.1,

$$\begin{aligned} \tilde{U}_{n+l}|_{B_j=A_j} &= U_{n+l}|_{T_1=\dots=T_l=r=0, B_j=A_j} \\ &= \frac{\partial}{\partial D_{l+n}} p_{l+n, N+1}(D_1, \dots, D_{l+n})|_{D_j=A_j} \\ &= (-1)^{l+n+1} (N+1) h_{l+n, N+1-l-n}(A_1, \dots, A_n, 0, \dots, 0) \\ &= (-1)^{l+n+1} (N+1) h_{N+1-l-n}(\mathbb{A}). \end{aligned}$$

Using Lemma 7.13, it is easy to see that

$$C(\Gamma)/\mathfrak{I} \cdot C(\Gamma) \simeq \begin{pmatrix} h_{N+1-l-n}(\mathbb{A}) & 0 \\ 0 & -A_1 - Y_1 \\ \cdots & \cdots \\ 0 & -A_n Y_{m-n} \\ 0 & 0_1 \\ \cdots & \cdots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{Y}|\mathbb{A})} \quad \{q^{-ln-(l+n-1)(m-n)}\}.$$

Next we exclude Y_1, \dots, Y_{m-n} by applying Proposition 2.19 to the second row through the $(m-n+1)$ -th row. This gives the relation

$$Y_j = (-1)^j h_j(\mathbb{A}) \text{ for } j = 0, 1, \dots, m-n,$$

and

$$C(\Gamma)/\mathfrak{I} \cdot C(\Gamma) \simeq \begin{pmatrix} h_{N+1-l-n}(\mathbb{A}) & 0 \\ 0 & -\sum_{j=0}^{m-n} (-1)^j h_j(\mathbb{A}) A_{m-n+1-j} \\ \cdots & \cdots \\ 0 & -\sum_{j=0}^{m-n} (-1)^j h_j(\mathbb{A}) A_{k-j} \\ \cdots & \cdots \\ 0 & -(-1)^{m-n} h_{m-n}(\mathbb{A}) A_n \\ 0 & 0_1 \\ \cdots & \cdots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})} \quad \{q^{-ln-(l+n-1)(m-n)}\}.$$

By equation (4.1), we have that, for $k = m-n+1, \dots, m$

$$\sum_{j=0}^{m-n} (-1)^j h_j(\mathbb{A}) A_{k-j} = - \sum_{j=m-n+1}^k (-1)^j h_j(\mathbb{A}) A_{k-j}.$$

So, using Corollary 2.16 and Lemma 7.13, we have

$$C(\Gamma)/\mathfrak{I} \cdot C(\Gamma) \simeq \begin{pmatrix} h_{N+1-l-n}(\mathbb{A}) & 0 \\ 0 & h_{m-n+1}(\mathbb{A}) \\ \cdots & \cdots \\ 0 & h_m(\mathbb{A}) \\ 0 & 0_1 \\ \cdots & \cdots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})} \quad \{q^{-ln-(l+n-1)(m-n)}\}.$$

If $m+l \leq N$, then $N+1-l-n \geq m-n+1$ and, therefore, $h_{N+1-l-n}(\mathbb{A})$ is in the ideal $(h_{m-n+1}(\mathbb{A}), \dots, h_m(\mathbb{A}))$. So, by Corollary 2.15,

$$C(\Gamma)/\mathfrak{I} \cdot C(\Gamma) \simeq \begin{pmatrix} 0 & 0_{n+l} \\ 0 & h_{m-n+1}(\mathbb{A}) \\ \cdots & \cdots \\ 0 & h_m(\mathbb{A}) \\ 0 & 0_1 \\ \cdots & \cdots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})} \quad \{q^{-ln-(l+n-1)(m-n)}\}.$$

Thus, by Proposition 2.20,

$$H(C(\Gamma)/\mathfrak{I} \cdot C(\Gamma)) \cong \begin{pmatrix} 0 & 0_{n+l} \\ 0 & 0_1 \\ \dots & \dots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})/(h_{m-n+1}(\mathbb{A}), \dots, h_m(\mathbb{A}))} \{q^{-ln-(l+n-1)(m-n)}\}.$$

Since the graded dimension of $\text{Sym}(\mathbb{A})/(h_{m-n+1}(\mathbb{A}), \dots, h_m(\mathbb{A}))$ is $\begin{bmatrix} m \\ n \end{bmatrix} \cdot q^{n(m-n)}$, it follows that

$$\text{gdim} C(\Gamma) = q^{-lm+m-n} \begin{bmatrix} m \\ n \end{bmatrix} (1 + \tau q^{2n+2l-N-1}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}).$$

If $m+l = N+1$, then $N+1-l-n = m-n$. Note that $h_m(\mathbb{A})$ is in the ideal $(h_{m-n}(\mathbb{A}), \dots, h_{m-1}(\mathbb{A}))$. By Lemma 2.13 and Corollary 2.16, we have

$$\begin{aligned} C(\Gamma)/\mathfrak{I} \cdot C(\Gamma) &\simeq \begin{pmatrix} 0 & h_{m-n}(\mathbb{A}) \\ 0 & h_{m-n+1}(\mathbb{A}) \\ \dots & \dots \\ 0 & h_m(\mathbb{A}) \\ 0 & 0_1 \\ \dots & \dots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})} \{q^{-ln-(l+n-1)(m-n)+N+1-2(m-n)}\} \langle 1 \rangle \\ &\simeq \begin{pmatrix} 0 & h_{m-n}(\mathbb{A}) \\ \dots & \dots \\ 0 & h_{m-1}(\mathbb{A}) \\ 0 & 0_m \\ 0 & 0_1 \\ \dots & \dots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})} \{q^{-ln-(l+n+1)(m-n)+N+1}\} \langle 1 \rangle. \end{aligned}$$

Thus, by Proposition 2.20,

$$H(C(\Gamma)/\mathfrak{I} \cdot C(\Gamma)) \simeq \begin{pmatrix} 0 & 0_m \\ 0 & 0_1 \\ \dots & \dots \\ 0 & 0_{m+l-1} \end{pmatrix}_{\text{Sym}(\mathbb{A})/(h_{m-n}(\mathbb{A}), \dots, h_{m-1}(\mathbb{A}))} \{q^{-ln-(l+n+1)(m-n)+N+1}\} \langle 1 \rangle.$$

Since the graded dimension of $\text{Sym}(\mathbb{A})/(h_{m-n}(\mathbb{A}), \dots, h_{m-1}(\mathbb{A}))$ is $\begin{bmatrix} m-1 \\ n \end{bmatrix} \cdot q^{n(m-n-1)}$, it follows that

$$\begin{aligned} &\text{gdim} C(\Gamma) \\ &= \tau q^{-ln-(l+n+1)(m-n)+N+1+n(m-n-1)} \begin{bmatrix} m-1 \\ n \end{bmatrix} (1 + \tau q^{2m-N-1}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}) \\ &= q^{-lm+m} \begin{bmatrix} m-1 \\ n \end{bmatrix} (1 + \tau q^{N+1-2m}) \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1}). \end{aligned}$$

Finally, let us consider equation (9.8).

Assume $m+l = N+1$. Then $\text{gdim} C(\Gamma_1) = 0$ and it is straightforward to see that $\text{gdim} C(\Gamma) = \begin{bmatrix} m-1 \\ n \end{bmatrix} \text{gdim} C(\Gamma_0)$. So (9.8) is true.

Assume $m + l \leq N$. Note that

$$\begin{bmatrix} m \\ n \end{bmatrix} = q^{-n} \begin{bmatrix} m-1 \\ n \end{bmatrix} + q^{m-n} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} = q^n \begin{bmatrix} m-1 \\ n \end{bmatrix} + q^{-m+n} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}.$$

So

$$\begin{aligned} & \begin{bmatrix} m \\ n \end{bmatrix} (1 + \tau q^{2n+2l-N-1}) \\ = & (q^n \begin{bmatrix} m-1 \\ n \end{bmatrix} + q^{-m+n} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}) + \tau q^{2n+2l-N-1} (q^{-n} \begin{bmatrix} m-1 \\ n \end{bmatrix} + q^{m-n} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}) \\ = & q^n \begin{bmatrix} m-1 \\ n \end{bmatrix} (1 + \tau q^{2l-N-1}) + q^{-m+n} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} (1 + \tau q^{2m+2l-N-1}). \end{aligned}$$

Multiplying $q^{-lm+m-n} \prod_{j=1}^{m+l-1} (1 + \tau q^{2j-N-1})$ to this equation, we get (9.8). \square

9.5. Proof of Theorem 9.1. With all the above preparations, we are now ready to prove Theorem 9.1.

Lemma 9.17. *Let Γ , Γ_0 and Γ_1 be the MOY graphs in Figure 43, where l, m, n are integers satisfying $0 \leq n \leq m \leq N$ and $0 \leq l, m + l - 1 \leq N$. Then there exist homogeneous morphisms*

$$\begin{aligned} \Phi &: C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\} \rightarrow C(\Gamma), \\ \Psi &: C(\Gamma) \rightarrow C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\} \end{aligned}$$

preserving both gradings such that

$$\Psi \circ \Phi \simeq \text{id}_{C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\}}.$$

Proof. Let $F, G, \vec{\alpha}, \vec{\beta}$ be the morphisms defined in Subsections 9.1 and 9.2. Define

$$\begin{aligned} \Phi_0 &: C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\} \rightarrow C(\Gamma), \\ \Psi_0 &: C(\Gamma) \rightarrow C(\Gamma_0) \left\{ \begin{bmatrix} m-1 \\ n \end{bmatrix} \right\} \oplus C(\Gamma_1) \left\{ \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \right\} \end{aligned}$$

by $\Phi_0 = (F, \vec{\alpha})$ and $\Psi_0 = (G, \vec{\beta})^T$. Then

$$\Psi_0 \circ \Phi_0 \simeq \begin{pmatrix} \text{id} & G \circ \vec{\alpha} \\ \vec{\beta} \circ F & \text{id} \end{pmatrix}.$$

Since $\vec{\beta} \circ F \circ G \circ \vec{\alpha}$ and $G \circ \vec{\alpha} \circ \vec{\beta} \circ F$ are homotopically nilpotent, $\text{id} - \vec{\beta} \circ F \circ G \circ \vec{\alpha}$ and $\text{id} - G \circ \vec{\alpha} \circ \vec{\beta} \circ F$ are homotopically invertible. In fact, their homotopical inverses are

$$\begin{aligned} (\text{id} - \vec{\beta} \circ F \circ G \circ \vec{\alpha})^{-1} &\simeq \sum_{k=0}^{\infty} (\vec{\beta} \circ F \circ G \circ \vec{\alpha})^k, \\ (\text{id} - G \circ \vec{\alpha} \circ \vec{\beta} \circ F)^{-1} &\simeq \sum_{k=0}^{\infty} (G \circ \vec{\alpha} \circ \vec{\beta} \circ F)^k. \end{aligned}$$

Note that the sums on the right hand side are finite sums in the Hom_{HMF} . Now define

$$\begin{aligned}\Phi &= \Phi_0, \\ \Psi &= \begin{pmatrix} (\text{id} - G \circ \vec{\alpha} \circ \vec{\beta} \circ F)^{-1} & 0 \\ 0 & (\text{id} - \vec{\beta} \circ F \circ G \circ \vec{\alpha})^{-1} \end{pmatrix} \circ \begin{pmatrix} \text{id} & -G \circ \vec{\alpha} \\ -\vec{\beta} \circ F & \text{id} \end{pmatrix} \circ \Psi_0.\end{aligned}$$

It is easy to check that Φ and Ψ satisfy all the requirements in the lemma. \square

Proof of Theorem 9.1. By Lemma 9.17 and Lemma 3.14, we know that there exists a graded matrix factorization M such that

$$C(\Gamma) \simeq C(\Gamma_0)\left\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\right\} \oplus C(\Gamma_1)\left\{\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right\} \oplus M.$$

But, by Lemma 9.16,

$$\text{gdim} M = \text{gdim} C(\Gamma) - \begin{bmatrix} m-1 \\ n \end{bmatrix} \cdot \text{gdim} C(\Gamma_0) - \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \cdot \text{gdim} C(\Gamma_1) = 0.$$

Thus, by Corollary 3.9, $M \simeq 0$. So

$$C(\Gamma) \simeq C(\Gamma_0)\left\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\right\} \oplus C(\Gamma_1)\left\{\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\right\}.$$

\square

10. DIRECT SUM DECOMPOSITION (V)

The main objective of this section is to prove Theorem 10.1, which “categorifies” [29, Proposition A.10] and further generalizes direct sum decomposition (IV) in [18]. The proof of Decomposition (V) is different from that of Decompositions (I–IV) in the sense that we do not explicitly construct the homotopy equivalence in Decomposition (V). Instead, we use the Krull-Schmidt property of the category hmf to prove this decomposition.

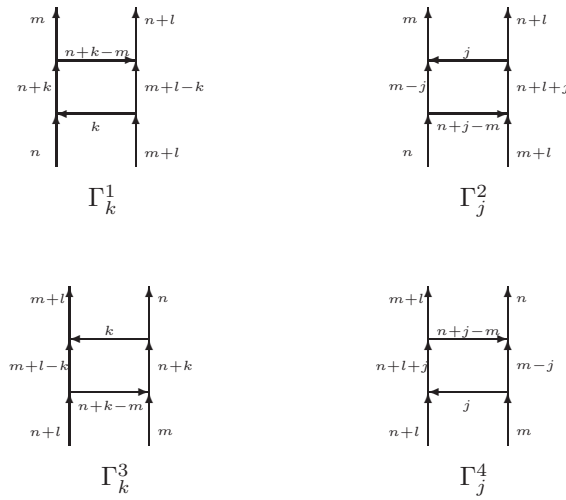


FIGURE 49.

Theorem 10.1. *Let m, n, l be non-negative integers satisfying $N \geq n + l, m + l$. For $\max\{m - n, 0\} \leq k \leq m + l$ and $\max\{m - n, 0\} \leq j \leq m$, define $\Gamma_k^1, \Gamma_k^3, \Gamma_j^2$ and Γ_j^4 to be the MOY graphs in Figure 49. Then, for $\max\{m - n, 0\} \leq k \leq m + l$,*

$$(10.1) \quad C(\Gamma_k^1) \simeq \bigoplus_{j=\max\{m-n,0\}}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k-j \end{bmatrix} \right\},$$

$$(10.2) \quad C(\Gamma_k^3) \simeq \bigoplus_{j=\max\{m-n,0\}}^m C(\Gamma_j^4) \left\{ \begin{bmatrix} l \\ k-j \end{bmatrix} \right\},$$

where we use the convention $\begin{bmatrix} a \\ b \end{bmatrix} = 0$ if $b < 0$ or $b > a$.

10.1. The proof. The $n \geq m$ case and the $n < m$ case of Theorem 10.1 may seem different. But, by flipping $\Gamma_k^1, \Gamma_k^3, \Gamma_j^2$ and Γ_j^4 horizontally and shifting the indices k, j , one can easily see that the $n \geq m$ (resp. $m \geq n$) case of equation (10.1) is equivalent to the $m \geq n$ (resp. $n \geq m$) case of equation (10.2). So, without loss of generality, we prove Theorem 10.1 under the assumption $n \geq m$.

We prove Theorem 10.1 by inducting on k . If $k = 0$, the decompositions (10.1) and (10.2) are trivially true. We prove the $k = 1$ case in the following lemma.

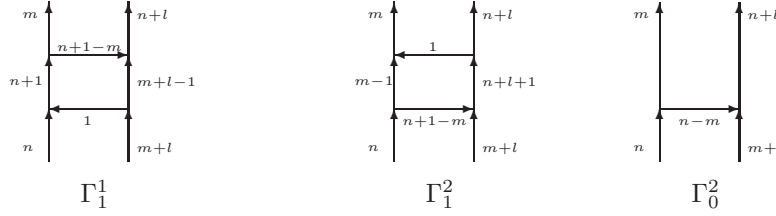


FIGURE 50.

Lemma 10.2. *Let $\Gamma_k^1, \Gamma_k^3, \Gamma_j^2$ and Γ_j^4 to be as in Theorem 10.1. Assume that $n \geq m$. Then*

$$(10.3) \quad C(\Gamma_1^1) \simeq C(\Gamma_1^2) \oplus C(\Gamma_0^2) \{[l]\},$$

$$(10.4) \quad C(\Gamma_1^3) \simeq C(\Gamma_1^4) \oplus C(\Gamma_0^4) \{[l]\}.$$

Proof. The proofs of (10.3) and (10.4) are very similar. So we only prove (10.3) here and leave (10.4) to the reader.

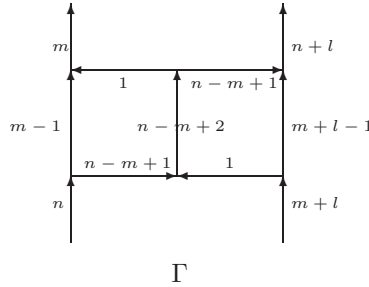


FIGURE 51.

Consider the MOY graph Γ in Figure 51. Applying Decomposition (IV) (Theorem 9.1) to the left square in Γ , we get $C(\Gamma) \simeq C(\Gamma_1^1) \oplus C(\Gamma')\{[m-1]\}$, where Γ' is given in Figure 52. By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have $C(\Gamma') \simeq C(\Gamma'') \simeq C(\Gamma_0^2)\{[m+l]\}$. Thus,

$$(10.5) \quad C(\Gamma) \simeq C(\Gamma_1^1) \oplus C(\Gamma_0^2)\{[m-1][m+l]\}.$$



FIGURE 52.

Now apply Decomposition (IV) (Theorem 9.1) to the right square in Γ . This gives $C(\Gamma) \simeq C(\Gamma_1^2) \oplus C(\Gamma''')\{[m+l-1]\}$, where Γ''' is given in Figure 53. By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have $C(\Gamma''') \simeq C(\Gamma''') \simeq C(\Gamma_0^2)\{[m]\}$. Thus,

$$(10.6) \quad C(\Gamma) \simeq C(\Gamma_1^2) \oplus C(\Gamma_0^2)\{[m][m+l-1]\}.$$

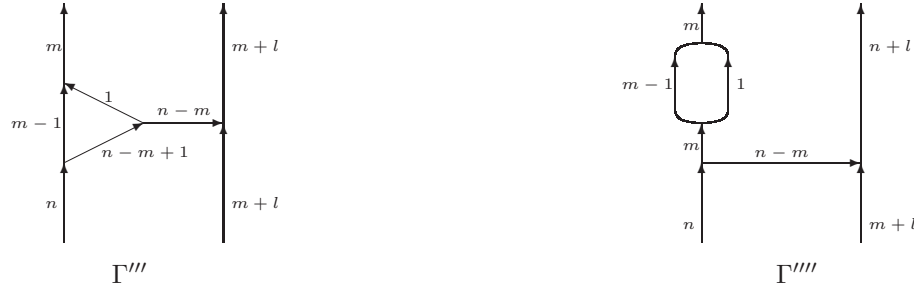


FIGURE 53.

Note that $[m] \cdot [m+l-1] - [m-1] \cdot [m+l] = [l]$. So, by the Krull-Schmidt property of the category hmf (Proposition 3.16 and Lemma 3.17), (10.5) and (10.6) imply (10.3). \square

With the above initial case in hand, we are ready to prove Theorem 10.1 in general.

Proof of Theorem 10.1. From above, we know that (10.1) and (10.2) are true for $k = 0, 1$. Now assume (10.1) and (10.2) are true for a given $k \geq 1$ and all m, n, l satisfying the conditions in Theorem 10.1. We claim that (10.1) and (10.2) are also true for $k+1$. The proofs for the $k+1$ cases of (10.1) and (10.2) are very similar. We only proof (10.1) for $k+1$ here and leave (10.2) to the reader.

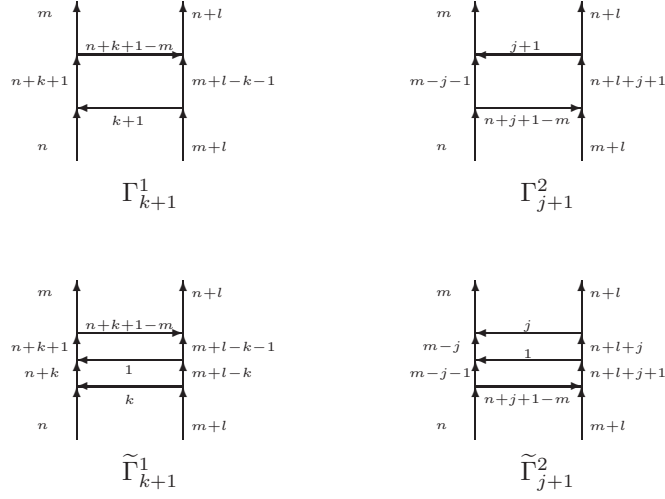


FIGURE 54.

Recall that Γ_{k+1}^1 and Γ_{j+1}^2 are the MOY graphs in the first row of Figure 54. We define $\tilde{\Gamma}_{k+1}^1$ and $\tilde{\Gamma}_{j+1}^2$ to be the MOY graphs in the second row in Figure 54. By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have

$$\begin{aligned} C(\tilde{\Gamma}_{k+1}^1) &\simeq C(\Gamma_{k+1}^1)\{[k+1]\}, \\ C(\tilde{\Gamma}_{j+1}^2) &\simeq C(\Gamma_{j+1}^2)\{[j+1]\}. \end{aligned}$$

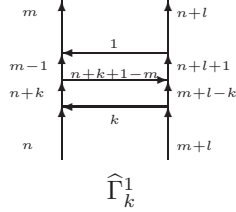


FIGURE 55.

Case 1. $k \leq l$. Apply (10.3) to the upper rectangle in $\tilde{\Gamma}_{k+1}^1$. This gives

$$C(\tilde{\Gamma}_{k+1}^1) \simeq C(\hat{\Gamma}_k^1) \oplus C(\Gamma_k^1)\{[l-k]\},$$

where $\hat{\Gamma}_k^1$ is the MOY graph in Figure 55 and Γ_k^1 is given in Figure 49. Recall that we assume (10.1) is true for the given k and **all** m, n, l satisfying the conditions in Theorem 10.1. Thus, we can apply (10.1) to the lower rectangle in $\hat{\Gamma}_k^1$ and get

$$\begin{aligned} C(\hat{\Gamma}_k^1) &\simeq \bigoplus_{j=0}^{m-1} C(\tilde{\Gamma}_{j+1}^2)\{[l+1] \atop [k-j]\} \\ &\simeq \bigoplus_{j=0}^{m-1} C(\Gamma_{j+1}^2)\{[j+1] \atop [k-j]\} \bigg[\begin{matrix} l+1 \\ k-j+1 \end{matrix} \bigg] = \bigoplus_{j=0}^m C(\Gamma_j^2)\{[j] \atop [k-j+1]\} \bigg[\begin{matrix} l+1 \\ k-j+1 \end{matrix} \bigg]. \end{aligned}$$

Again, recall that we assume (10.1) is true for Γ_k^1 . That is,

$$C(\Gamma_k^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k-j \end{bmatrix} \right\}.$$

Note that $[j] \begin{bmatrix} l+1 \\ k-j+1 \end{bmatrix} + [l-k] \begin{bmatrix} l \\ k-j \end{bmatrix} = \begin{bmatrix} l \\ k+1-j \end{bmatrix} [k+1]$. So, combining the above, we get

$$C(\Gamma_{k+1}^1) \{[k+1]\} \simeq C(\tilde{\Gamma}_{k+1}^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k+1-j \end{bmatrix} [k+1] \right\}.$$

By Proposition 3.20, this implies

$$C(\Gamma_{k+1}^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k+1-j \end{bmatrix} \right\}.$$

So (10.1) is true for $k+1$ if $k \leq l$.

Case 2. $k > l$. In this case, we apply (10.4) to the upper rectangle of $\hat{\Gamma}_k^1$. This gives

$$C(\hat{\Gamma}_k^1) \simeq C(\tilde{\Gamma}_{k+1}^1) \oplus C(\Gamma_k^1) \{[k-l]\}.$$

Note that, in this case, we also have

$$C(\hat{\Gamma}_k^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ [j] \begin{bmatrix} l+1 \\ k-j+1 \end{bmatrix} \right\}$$

and

$$C(\Gamma_k^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k-j \end{bmatrix} \right\}.$$

Note that $[j] \begin{bmatrix} l+1 \\ k-j+1 \end{bmatrix} - [k-l] \begin{bmatrix} l \\ k-j \end{bmatrix} = \begin{bmatrix} l \\ k+1-j \end{bmatrix} [k+1]$. So, by Lemma 3.17, we have

$$C(\Gamma_{k+1}^1) \{[k+1]\} \simeq C(\tilde{\Gamma}_{k+1}^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k+1-j \end{bmatrix} [k+1] \right\}.$$

By Proposition 3.20, this implies

$$C(\Gamma_{k+1}^1) \simeq \bigoplus_{j=0}^m C(\Gamma_j^2) \left\{ \begin{bmatrix} l \\ k+1-j \end{bmatrix} \right\}.$$

So (10.1) is true for $k+1$ if $k > l$. □

11. CHAIN COMPLEXES ASSOCIATED TO KNOTTED MOY GRAPHS



FIGURE 56.

Definition 11.1. A knotted MOY graph is an immersion of an abstract MOY graph into \mathbb{R}^2 such that

- the only singularities are finitely many transversal double points in the interior of edges (i.e. away from the vertices),
- we specify the upper edge and the lower edge at each of these transversal double points.

Each transversal double point in a knotted MOY graph is called a crossing. We follow the usual sign convention for crossings given in Figure 56.

If there are crossings in an edge, these crossing divide the edge into several parts. We call each part a segment of the edge.

Note that colored oriented link/tangle diagrams and (embedded) MOY graphs are special cases of knotted MOY graphs.

Definition 11.2. A marking of a knotted MOY graph D consists the following:

- (1) A finite collection of marked points on D such that
 - every segment of every edge of D has at least one marked point;
 - all the end points (vertices of valence 1) are marked;
 - none of the crossings and interior vertices (vertices of valence at least 2) is marked.
- (2) An assignment of pairwise disjoint alphabets to the marked points such that the alphabet associated to a marked point on an edge of color m has m independent indeterminants. (Recall that an alphabet is a finite collection of homogeneous indeterminants of degree 2.)

Given a knotted MOY graph D with a marking, we cut D open at the marked points. This produces a collection $\{D_1, \dots, D_m\}$ of simple knotted MOY graphs marked only at their end points. We call each D_i a piece of D . It is easy to see that each D_i is one of the following:

- (i) an oriented arc from one marked point to another,
- (ii) a star-shaped neighborhood of a vertex in an (embedded) MOY graph,
- (iii) a crossing with colored branches.

For a given D_i , let $\mathbb{X}_1, \dots, \mathbb{X}_{n_i}$ be the alphabets assigned to all end points of D_i , among which $\mathbb{X}_1, \dots, \mathbb{X}_{k_i}$ are assigned to exits and $\mathbb{X}_{k_i+1}, \dots, \mathbb{X}_{n_i}$ are assigned to entrances. Let $R_i = \text{Sym}(\mathbb{X}_1 | \dots | \mathbb{X}_{n_i})$ and $w_i = \sum_{j=1}^{k_i} p_{N+1}(\mathbb{X}_j) - \sum_{j=k_i+1}^{n_i} p_{N+1}(\mathbb{X}_j)$. Then the chain complex $C(D_i)$ associated to D_i is an object of $\text{hCh}^b(\text{hmf}_{R_i, w_i})$.

If D_i is of type (i) or (ii), then it is an (embedded) MOY graph, and its matrix factorization $C(D_i)$ is an object of hmf_{R_i, w_i} . We define the chain complex associated to D_i , which is denoted by $\hat{C}(D_i) = C(D_i)$, to be

$$0 \rightarrow C(D_i) \rightarrow 0,$$

where $C(D_i)$ has homological grading 0. (The abuse of notations here should not be confusing.)

If D_i is of type (iii), i.e. a colored crossing, the definitions of $\hat{C}(D_i)$ and $C(D_i)$ are much more complex. The chain complexes associated to colored crossings will be defined in Definition 11.16 below.

Remark 11.3. In the present paper, $\hat{C}(*)$ stands for the unnormalized chain complex of $*$ and $C(*)$ stands for the normalized chain complex of $*$. For pieces of types (i) and (ii), there is no difference between their normalized and unnormalized chain complexes. For a piece of type (iii), i.e. a colored crossing, these two complexes differ by a shift in all three gradings. See Definition 11.16 below for details.

Once we defined the chain complexes associated to each piece D_i , then we can define the chain complex associated to D .

Definition 11.4.

$$\begin{aligned}\hat{C}(D) &:= \bigotimes_{i=1}^m \hat{C}(D_i), \\ C(D) &:= \bigotimes_{i=1}^m C(D_i),\end{aligned}$$

where the tensor product is done over the common end points. For example, for two pieces D_{i_1} and D_{i_2} of D , let $\mathbb{W}_1, \dots, \mathbb{W}_l$ be the alphabets associated to their common end points. Then, in the above tensor product,

$$C(D_{i_1}) \otimes C(D_{i_2}) = C(D_{i_1}) \otimes_{\text{Sym}(\mathbb{W}_1 | \dots | \mathbb{W}_l)} C(D_{i_2}).$$

If D is closed, i.e. has no endpoints, then $C(D)$ is an object of $\mathbf{hCh}^b(\mathbf{hmf}_{C,0})$. Assume D has endpoints. Let $\mathbb{E}_1, \dots, \mathbb{E}_n$ be the alphabets assigned to all end points of D , among which $\mathbb{E}_1, \dots, \mathbb{E}_k$ are assigned to exits and $\mathbb{E}_{k+1}, \dots, \mathbb{E}_n$ are assigned to entrances. Let $R = \text{Sym}(\mathbb{E}_1 | \dots | \mathbb{E}_n)$ and $w = \sum_{i=1}^k p_{N+1}(\mathbb{E}_i) - \sum_{j=k+1}^n p_{N+1}(\mathbb{E}_j)$. In this case, $C(D)$ is an object of $\mathbf{hCh}^b(\mathbf{hmf}_{R,w})$.

Note that, as an object of $\mathbf{hCh}^b(\mathbf{hmf}_{R,w})$, $C(D)$ has a \mathbb{Z}_2 -grading, a quantum grading and a homological grading.

In the rest of this section, we define and study the chain complexes associated to colored crossings. For this purpose, we need to understand the morphisms between matrix factorizations associated to MOY graphs of the type shown in Figure 57.

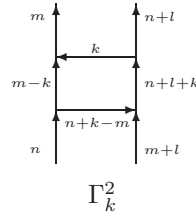


FIGURE 57.

11.1. Change of base ring. There is a change of base ring involved in the computation of $\text{Hom}_{\mathbf{HMF}}(C(\Gamma_k^2), *)$, which is the subject of this subsection.

Let $\mathbb{A} = \{a_1, \dots, a_m\}$, $\mathbb{B} = \{b_1, \dots, b_n\}$ and $\mathbb{X} = \{x_1, \dots, x_{m+n}\}$ be alphabets. Denote by A_k , B_k and X_k the k -th elementary symmetric polynomials in \mathbb{A} , \mathbb{B} and \mathbb{X} . Define

$$\begin{aligned}E_k &= X_k - \sum_{j=0}^k A_j B_{k-j}, \\ H_k &= \sum_{j=0}^k (-1)^j h_j(\mathbb{A}) X_{k-j} - B_k \\ &= \begin{cases} \sum_{j=0}^k (-1)^j h_j(\mathbb{A}) X_{k-j} - B_k & \text{if } k = 0, 1, \dots, n, \\ \sum_{j=0}^k (-1)^j h_j(\mathbb{A}) X_{k-j} & \text{if } k = n+1, \dots, n+m. \end{cases}\end{aligned}$$

Define I_1 and I_2 to be the homogeneous ideals of $\text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{X})$ given by

$$\begin{aligned} I_1 &= (E_1, \dots, E_{m+n}), \\ I_2 &= (H_1, \dots, H_{m+n}). \end{aligned}$$

Lemma 11.5. $I_1 = I_2$.

Proof. First, note that

$$\begin{aligned} \sum_{i=0}^k (-1)^i h_i(\mathbb{A}) E_{k-i} &= \sum_{i=0}^k (-1)^i h_i(\mathbb{A}) X_{k-i} - \sum_{i=0}^k \sum_{j=0}^{k-i} (-1)^i h_i(\mathbb{A}) A_{k-i-j} B_j \\ &= \sum_{i=0}^k (-1)^i h_i(\mathbb{A}) X_{k-i} - \sum_{j=0}^k B_j \sum_{i=0}^{k-j} (-1)^i h_i(\mathbb{A}) A_{k-i-j} \\ &\quad \text{(by equation (4.1))} = \sum_{i=0}^k (-1)^i h_i(\mathbb{A}) X_{k-i} - B_k = H_k. \end{aligned}$$

This shows that $I_2 \subset I_1$.

Next, we have

$$\begin{aligned} \sum_{i=0}^k A_i H_{k-i} &= \sum_{i=0}^k A_i \sum_{j=0}^{k-i} (-1)^{k-i-j} h_{k-i-j}(\mathbb{A}) X_j - \sum_{i=0}^k A_i B_{k-i} \\ &= \sum_{j=0}^k X_j \sum_{i=0}^{k-j} (-1)^{k-i-j} h_{k-i-j}(\mathbb{A}) A_i - \sum_{i=0}^k A_i B_{k-i} \\ &\quad \text{(by equation (4.1))} = X_k - \sum_{i=0}^k A_i B_{k-i} = E_k. \end{aligned}$$

So $I_1 \subset I_2$. Altogether, we have $I_1 = I_2$. \square

Note that, for $k = n+1, \dots, n+m$, $H_k \in \text{Sym}(\mathbb{A}|\mathbb{X})$. Define I_3 to be the homogeneous ideal of $\text{Sym}(\mathbb{A}|\mathbb{X})$ given by $I_3 = (H_{n+1}, \dots, H_{n+m})$.

Lemma 11.6. *The quotient ring $\text{Sym}(\mathbb{A}|\mathbb{X})/I_3$ is a finitely generated graded-free $\text{Sym}(\mathbb{X})$ -module of graded rank $\begin{bmatrix} m+n \\ n \end{bmatrix}$.*

As graded $\text{Sym}(\mathbb{A}|\mathbb{X})/I_3$ -modules, we have

$$(11.1) \quad \text{Hom}_{\text{Sym}(\mathbb{X})}(\text{Sym}(\mathbb{A}|\mathbb{X})/I_3, \text{Sym}(\mathbb{X})) \cong \text{Sym}(\mathbb{A}|\mathbb{X})/I_3 \{q^{-2mn}\}.$$

Proof. Note that

$$\text{Sym}(\mathbb{A}|\mathbb{X})/I_3 \cong \text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{X})/I_2 \cong \text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{X})/I_1,$$

where the isomorphisms preserve both the graded ring structure and the graded $\text{Sym}(\mathbb{X})$ -module structure.

By Theorem 4.3, $\text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{X})/I_1$ is a finitely generated graded-free $\text{Sym}(\mathbb{X})$ -module of graded rank $\begin{bmatrix} m+n \\ n \end{bmatrix}$. From the above isomorphism, so is $\text{Sym}(\mathbb{A}|\mathbb{X})/I_3$. Moreover, by Theorem 4.3, there is a Sylvester operator on $\text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{X})/I_1$ and a pair of homogeneous $\text{Sym}(\mathbb{X})$ -basis of $\text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{X})/I_1$ that are duals of each other under the Sylvester operator. These induce a pair of homogeneous $\text{Sym}(\mathbb{X})$ -basis $\{S_\lambda | \lambda \in \Lambda_{m,n}\}$ and $\{S'_\lambda | \lambda \in \Lambda_{m,n}\}$ of $\text{Sym}(\mathbb{A}|\mathbb{X})/I_3$ and a Sylvester operator

$$\zeta : \text{Sym}(\mathbb{A}|\mathbb{X})/I_3 \rightarrow \text{Sym}(\mathbb{X})$$

such that, for $\lambda, \mu \in \Lambda_{m,n}$,

$$\zeta(S_\lambda \cdot S'_\mu) = \begin{cases} 1 & \text{if } \mu = \lambda^c, \\ 0 & \text{if } \mu \neq \lambda^c. \end{cases}$$

(Recall that $\Lambda_{m,n} = \{\lambda = (\lambda_1 \geq \dots \geq \lambda_m) \mid l(\lambda) \leq m, \lambda_1 \leq n\}$, and $\lambda^c = (n - \lambda_m \geq \dots \geq n - \lambda_1)$.)

One can see from the above that $\{\zeta(S_\lambda \cdot *) \mid \lambda \in \Lambda_{m,n}\}$ is the $\text{Sym}(\mathbb{X})$ -basis of $\text{Hom}_{\text{Sym}(\mathbb{X})}(\text{Sym}(\mathbb{A}|\mathbb{X})/I_3, \text{Sym}(\mathbb{X}))$ dual to $\{S'_\lambda \mid \lambda \in \Lambda_{m,n}\}$. So the $\text{Sym}(\mathbb{X})$ -module map

$$\text{Sym}(\mathbb{A}|\mathbb{X})/I_3 \rightarrow \text{Hom}_{\text{Sym}(\mathbb{X})}(\text{Sym}(\mathbb{A}|\mathbb{X})/I_3, \text{Sym}(\mathbb{X}))$$

given by $u \mapsto \zeta(u \cdot *)$ is a homogeneous isomorphism of $\text{Sym}(\mathbb{X})$ -modules of degree $-2mn$. It is easy to see that this map is also $\text{Sym}(\mathbb{A}|\mathbb{X})/I_3$ -linear. This proves (11.1). \square

Lemma 11.7. *Let $\mathbb{A} = \{a_1, \dots, a_m\}$, $\mathbb{X} = \{x_1, \dots, x_{m+n}\}$, $\mathbb{Y}_1, \dots, \mathbb{Y}_k$ be alphabets. Define*

$$\begin{aligned} R &= \text{Sym}(\mathbb{A}|\mathbb{X}|\mathbb{Y}_1|\dots|\mathbb{Y}_k)/(H_{n+1}, \dots, H_{n+m}), \\ \hat{R} &= \text{Sym}(\mathbb{X}|\mathbb{Y}_1|\dots|\mathbb{Y}_k), \end{aligned}$$

where H_{n+1}, \dots, H_{n+m} are the polynomials in $\text{Sym}(\mathbb{A}|\mathbb{X})$ given above. Then \hat{R} is a subring of R through the composition of the standard inclusion and projection $\hat{R} \hookrightarrow \text{Sym}(\mathbb{A}|\mathbb{X}|\mathbb{Y}_1|\dots|\mathbb{Y}_k) \rightarrow R$.

Suppose that w is a homogeneous element of \hat{R} of degree $2(N+1)$ and M is a finitely generated graded matrix factorization over R with potential w . Then, $\text{Hom}_{\hat{R}}(M, \hat{R})$ and $\text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R}))$ are both graded matrix factorizations over R of potential $-w$. Moreover, as graded matrix factorizations over R ,

$$(11.2) \quad \text{Hom}_{\hat{R}}(M, \hat{R}) \cong \text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R})) \cong \text{Hom}_R(M, R)\{q^{-2mn}\}.$$

Proof. Recall that the R -module structures on $\text{Hom}_{\hat{R}}(M, \hat{R})$ and $\text{Hom}_{\hat{R}}(R, \hat{R})$ are given by “multiplication on the inside”. From Lemma 11.6, we know that, as graded \hat{R} -modules, $R \cong \hat{R}\left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]$ and, as graded R -modules, $\text{Hom}_{\hat{R}}(R, \hat{R}) \cong R\{q^{-2mn}\}$. So $\text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R})) \cong \text{Hom}_R(M, R)\{q^{-2mn}\}$ is a graded matrix factorization over R of potential $-w$.

Define $\alpha : \text{Hom}_{\hat{R}}(M, \hat{R}) \rightarrow \text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R}))$ by $\alpha(f)(m)(r) = f(r \cdot m) \forall f \in \text{Hom}_{\hat{R}}(M, \hat{R}), m \in M, r \in R$. Define $\beta : \text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R})) \rightarrow \text{Hom}_{\hat{R}}(M, \hat{R})$ by $\beta(g)(m) = g(m)(1) \forall g \in \text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R})), m \in M$. It is straightforward to check that

- α and β are R -module isomorphisms and are inverses of each other.
- α and β preserve both the \mathbb{Z}_2 -grading and the quantum grading.

This implies that $\text{Hom}_{\hat{R}}(M, \hat{R})$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}$ -graded-free R -module isomorphic to $\text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R})) \cong \text{Hom}_R(M, R)\{q^{-2mn}\}$. It is easy to check that the differential of M induces on $\text{Hom}_{\hat{R}}(M, \hat{R})$ an R -linear differential making it a graded matrix factorization over R of potential $-w$.

To prove the lemma, it remains to check that α and β commute with the differentials of $\text{Hom}_{\hat{R}}(M, \hat{R})$ and $\text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R}))$. Since α and β are inverses of each other, we only need to show that α commutes with the differentials. Recall that, if $f \in \text{Hom}_{\hat{R}}(M, \hat{R})$ and $g \in \text{Hom}_R(M, \text{Hom}_{\hat{R}}(R, \hat{R}))$ have \mathbb{Z}_2 -degree ε , then

$df = (-1)^{\varepsilon+1}f \circ d_M$ and $dg = (-1)^{\varepsilon+1}g \circ d_M$. So, for any $f \in \text{Hom}_{\hat{R}}(M, \hat{R})$ with \mathbb{Z}_2 -degree ε and $m \in M$, $r \in R$, we have

$$\begin{aligned} \alpha(df)(m)(r) &= (df)(r \cdot m) = (-1)^{\varepsilon+1}f(d_M(r \cdot m)) = (-1)^{\varepsilon+1}f(r \cdot d_M(m)) \\ &= (-1)^{\varepsilon+1}\alpha(f)(d_M(m))(r) = d(\alpha(f))(m)(r). \end{aligned}$$

This shows that $\alpha \circ d = d \circ \alpha$. \square

11.2. Computing $\text{Hom}_{\text{HMF}}(C(\Gamma_k^2), *)$. Let Γ_k^2 be the MOY graph in Figure 57. We mark it as in Figure 58, where we omit the markings on the two horizontal edges since these are not explicitly used.

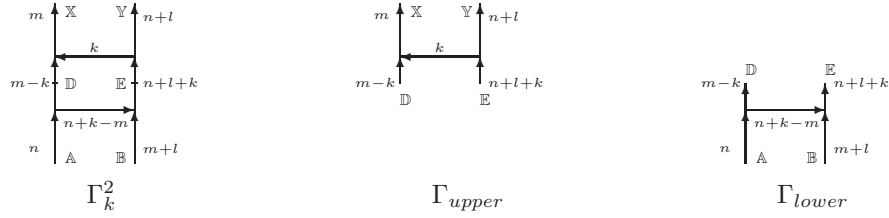


FIGURE 58.

Lemma 11.8.

$$C(\Gamma_k^2) \simeq \begin{pmatrix} * & S_1 + Y_1 - T_1 - B_1 \\ \dots & \dots \\ * & \sum_{i=0}^j (S_i Y_{j-i} - T_i B_{j-i}) \\ \dots & \dots \\ * & \sum_{i=0}^{n+l+k} (S_i Y_{n+l+k-i} - T_i B_{n+l+k-i}) \\ * & S_m \\ \dots & \dots \\ * & S_k + 1 \\ * & T_n \\ \dots & \dots \\ * & T_{n-k+m+1} \end{pmatrix}_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B}|\mathbb{D})} \{q^{-k(n+l)-(m-k)(n+k-m)}\},$$

where

$$\begin{aligned} S_j &= \sum_{i=0}^j (-1)^i h_i(\mathbb{D}) X_{j-i}, \\ T_j &= \sum_{i=0}^j (-1)^i h_i(\mathbb{D}) A_{j-i}, \end{aligned}$$

and X_j , Y_j , A_j , B_j , D_j , E_j are the j -th elementary symmetric polynomials in the corresponding alphabets.

Proof. Cutting Γ_k^2 open horizontally in the middle, we get the MOY graphs Γ_{upper} and Γ_{lower} in Figure 58. Applying Lemma 7.22 to Γ_{upper} , we get

$$C(\Gamma_{upper}) \simeq \left(\begin{array}{c} * \\ \dots \\ * \\ \dots \\ * \\ * \\ \dots \\ * \end{array} \begin{array}{l} S_1 + Y_1 - E_1 \\ \dots \\ (\sum_{i=0}^j S_i Y_{j-i}) - E_j \\ \dots \\ (\sum_{i=0}^{n+l+k} S_i Y_{n+l+k-i}) - E_{n+l+k} \\ S_m \\ \dots \\ S_k + 1 \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{D}|\mathbb{E})} \{q^{-k(n+l)}\}.$$

Applying Lemma 7.22 to Γ_{lower} , we get

$$C(\Gamma_{lower}) \simeq \left(\begin{array}{c} * \\ \dots \\ * \\ \dots \\ * \\ * \\ \dots \\ * \end{array} \begin{array}{l} E_1 - T_1 - B_1 \\ \dots \\ E_j - \sum_{i=0}^j T_i B_{j-i} \\ \dots \\ E_{n+l+k} - \sum_{i=0}^{n+l+k} T_i B_{n+l+k-i} \\ T_n \\ \dots \\ T_{n-k+m+1} \end{array} \right)_{\text{Sym}(\mathbb{A}|\mathbb{B}|\mathbb{D}|\mathbb{E})} \{q^{-(m-k)(n+k-m)}\},$$

Thus,

$$\begin{aligned} C(\Gamma_k^2) &\simeq C(\Gamma_{upper}) \otimes_{\text{Sym}(\mathbb{D}|\mathbb{E})} C(\Gamma_{lower}) \\ &\simeq \left(\begin{array}{c} * \\ \dots \\ * \\ \dots \\ * \\ * \\ \dots \\ * \\ * \\ \dots \\ * \\ * \\ \dots \\ * \\ * \\ \dots \\ * \end{array} \begin{array}{l} S_1 + Y_1 - E_1 \\ \dots \\ (\sum_{i=0}^j S_i Y_{j-i}) - E_j \\ \dots \\ (\sum_{i=0}^{n+l+k} S_i Y_{n+l+k-i}) - E_{n+l+k} \\ S_m \\ \dots \\ S_k + 1 \\ E_1 - T_1 - B_1 \\ \dots \\ E_j - \sum_{i=0}^j T_i B_{j-i} \\ \dots \\ E_{n+l+k} - \sum_{i=0}^{n+l+k} T_i B_{n+l+k-i} \\ T_n \\ \dots \\ T_{n-k+m+1} \end{array} \right)_{\text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B}|\mathbb{D}|\mathbb{E})} \{q^{-k(n+l)-(m-k)(n+k-m)}\}, \end{aligned}$$

From here, the lemma is obtained by excluding E_1, \dots, E_{n+l+k} from the base ring by applying Proposition 2.19 to the rows

$$\left(\begin{array}{c} * \\ \dots \\ * \\ \dots \\ * \end{array} \begin{array}{l} E_1 - T_1 - B_1 \\ \dots \\ E_j - \sum_{i=0}^j T_i B_{j-i} \\ \dots \\ E_{n+l+k} - \sum_{i=0}^{n+l+k} T_i B_{n+l+k-i} \end{array} \right)$$

in the above Koszul matrix factorization. \square

Lemma 11.9. *Let Γ_k^2 be the MOY graph in Figure 58, and $\overline{\Gamma_k^2}$ the MOY graph obtained by reversing the orientations of all edges of Γ_k^2 . Suppose that M is a matrix factorization over $\hat{R} := \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$ with potential*

$$w = p_{N+1}(\mathbb{X}) + p_{N+1}(\mathbb{Y}) - p_{N+1}(\mathbb{A}) - p_{N+1}(\mathbb{B}).$$

Then,

$$\text{Hom}_{\text{HMF}}(C(\Gamma_k^2), M) \cong H(M \otimes_{\hat{R}} C(\overline{\Gamma_k^2})) \langle m+n+l \rangle \{q^{(l+m+n)(N-l)-m^2-n^2}\},$$

where $H(M \otimes_{\hat{R}} C(\overline{\Gamma_k^2}))$ is the usual homology of the chain complex $M \otimes_{\hat{R}} C(\overline{\Gamma_k^2})$.

Proof. Let S_j and T_j be as in Lemma 11.8. Define

$$R = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B}|\mathbb{D}) / (S_{k+1}, \dots, S_m).$$

Let

$$\begin{aligned} \mathcal{M} &= \begin{pmatrix} * & S_1 + Y_1 - T_1 - B_1 \\ \dots & \dots \\ * & \sum_{i=0}^j (S_i Y_{j-i} - T_i B_{j-i}) \\ \dots & \dots \\ * & \sum_{i=0}^{n+l+k} (S_i Y_{n+l+k-i} - T_i B_{n+l+k-i}) \\ * & T_n \\ \dots & \dots \\ * & T_{n-k+m+1} \end{pmatrix}_R, \\ \overline{\mathcal{M}} &= \begin{pmatrix} * & -(S_1 + Y_1 - T_1 - B_1) \\ \dots & \dots \\ * & -\sum_{i=0}^j (S_i Y_{j-i} - T_i B_{j-i}) \\ \dots & \dots \\ * & -\sum_{i=0}^{n+l+k} (S_i Y_{n+l+k-i} - T_i B_{n+l+k-i}) \\ * & -T_n \\ \dots & \dots \\ * & -T_{n-k+m+1} \end{pmatrix}_R. \end{aligned}$$

Then

$$\text{Hom}_R(\mathcal{M}, R) \cong \overline{\mathcal{M}} \langle m+n+l \rangle \{q^{(m+n+l)(N+1)-\sum_{i=1}^{n+l+k} 2i - \sum_{j=n-k+m+1}^n 2j}\}.$$

By Lemma 11.8 and Proposition 2.22, we have

$$\begin{aligned} \text{Hom}_{\text{HMF}}(C(\Gamma_k^2), M) &:= H(\text{Hom}_{\hat{R}}(C(\Gamma_k^2), M)) \\ &= H(\text{Hom}_{\hat{R}}(\mathcal{M}\{q^{-k(n+l)-(m-k)(n+k-m)}\}, M)) \\ &= H(\text{Hom}_{\hat{R}}(\mathcal{M}, M))\{q^{k(n+l)+(m-k)(n+k-m)}\}. \end{aligned}$$

Note that \mathcal{M} is finitely generated over R and over \hat{R} . By Lemma 11.7, we have

$$\begin{aligned} \text{Hom}_{\hat{R}}(\mathcal{M}, M) &\cong M \otimes_{\hat{R}} \text{Hom}_{\hat{R}}(\mathcal{M}, \hat{R}) \\ &\cong M \otimes_{\hat{R}} \text{Hom}_R(\mathcal{M}, R)\{q^{-2k(m-k)}\}. \end{aligned}$$

Altogether, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma_k^2), M) &\cong H(\mathrm{Hom}_{\hat{R}}(\mathcal{M}, M))\{q^{k(n+l)+(m-k)(n+k-m)}\} \\ &\cong H(M \otimes_{\hat{R}} \mathrm{Hom}_R(\mathcal{M}, R))\{q^{k(n+l)+(m-k)(n+k-m)-2k(m-k)}\} \\ &\cong H(M \otimes_{\hat{R}} \overline{\mathcal{M}})\langle m+n+l \rangle \{q^\varsigma\}, \end{aligned}$$

where

$$\varsigma = k(n+l)+(m-k)(n+k-m)-2k(m-k)+(m+n+l)(N+1)-\sum_{i=1}^{n+l+k} 2i - \sum_{j=n-k+m+1}^n 2j.$$

On the other hand,

$$H(M \otimes_{\hat{R}} C(\overline{\Gamma_k^2})) \cong H(M \otimes_{\hat{R}} \overline{\mathcal{M}})\{q^{-k(m-k)-(m+l)(n+k-m)}\}.$$

So

$$\mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma_k^2), M) \cong H(M \otimes_{\hat{R}} C(\overline{\Gamma_k^2}))\langle m+n+l \rangle \{q^{\varsigma+k(m-k)+(m+l)(n+k-m)}\}.$$

One can check that

$$\varsigma + k(m-k) + (m+l)(n+k+m) = (l+m+n)(N-l) - m^2 - n^2.$$

This proves the lemma. \square

11.3. The chain complex associated to a colored crossing. Let $c_{m,n}^+$ and $c_{m,n}^-$ be the colored crossings with marked end points in Figure 59. In this subsection, we define the chain complexes associated to them, which completes the definition of chain complexes associated to knotted MOY graphs.



FIGURE 59.

For $\max\{m-n, 0\} \leq k \leq m$, we call Γ_k^L and Γ_k^R in Figure 60 the k -th left and right resolutions of $c_{m,n}^\pm$. The following lemma is a special case of Decomposition (V) (Theorem 10.1.)

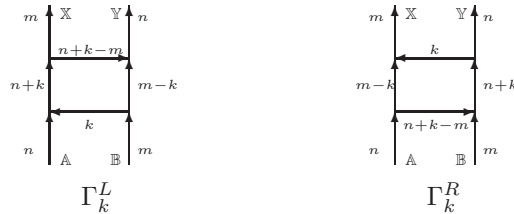


FIGURE 60.

Lemma 11.10. *Let m, n be integers such that $0 \leq m, n \leq N$. For $\max\{m-n, 0\} \leq k \leq m$, define Γ_k^L and Γ_k^R to be the MOY graphs in Figure 60. Then $C(\Gamma_k^L) \simeq C(\Gamma_k^R)$.*

Lemma 11.11. *Let m, n be integers such that $0 \leq m, n \leq N$. For $\max\{m-n, 0\} \leq j, k \leq m$,*

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\Gamma_j^L), C(\Gamma_k^L)) \cong \text{Hom}_{\text{HMF}}(C(\Gamma_j^R), C(\Gamma_k^L)) \\ & \cong \text{Hom}_{\text{HMF}}(C(\Gamma_j^L), C(\Gamma_k^R)) \cong \text{Hom}_{\text{HMF}}(C(\Gamma_j^R), C(\Gamma_k^R)) \\ & \cong C(\emptyset) \left\{ \begin{bmatrix} n+j+k-m \\ k \end{bmatrix} \begin{bmatrix} n+j+k-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-k \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-j \end{bmatrix} \begin{bmatrix} N \\ n+j+k-m \end{bmatrix} q^{(m+n)N-n^2-m^2} \right\}. \end{aligned}$$

In particular, the lowest non-vanishing quantum grading of these spaces are all $(k-j)^2$. And the subspaces of homogeneous elements of quantum degree $(k-j)^2$ of these spaces are 1-dimensional and have \mathbb{Z}_2 grading 0.

Proof. By Lemma 11.10, the above four Hom_{HMF} spaces are isomorphic. So, to prove the lemma, we only need to compute one of these, say $\text{Hom}_{\text{HMF}}(C(\Gamma_j^R), C(\Gamma_k^L))$.

Let $\hat{R} = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$. By Lemma 11.9,

$$\text{Hom}_{\text{HMF}}(C(\Gamma_j^R), C(\Gamma_k^L)) \cong H(C(\Gamma_k^L) \otimes_{\hat{R}} C(\overline{\Gamma_j^R})) \langle m+n \rangle \{ q^{(m+n)N-n^2-m^2} \},$$

where $\overline{\Gamma_j^R}$ is Γ_j^R with the opposite orientation.

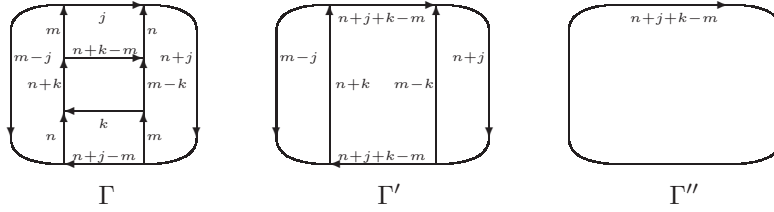


FIGURE 61.

Let Γ , Γ' and Γ'' be the MOY graphs in Figure 61. Then, by Corollary 5.13 and Decompositions (I-II) (Theorems 5.16 and 5.14), we have

$$\begin{aligned} & C(\Gamma_k^L) \otimes_{\hat{R}} C(\overline{\Gamma_j^R}) = C(\Gamma) \\ & \cong C(\Gamma') \left\{ \begin{bmatrix} n+j+k-m \\ k \end{bmatrix} \begin{bmatrix} n+j+k-m \\ j \end{bmatrix} \right\} \\ & \cong C(\Gamma'') \langle j+k \rangle \left\{ \begin{bmatrix} n+j+k-m \\ k \end{bmatrix} \begin{bmatrix} n+j+k-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-k \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-j \end{bmatrix} \right\} \\ & \cong C(\emptyset) \langle m+n \rangle \left\{ \begin{bmatrix} n+j+k-m \\ k \end{bmatrix} \begin{bmatrix} n+j+k-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-k \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-j \end{bmatrix} \begin{bmatrix} N \\ n+j+k-m \end{bmatrix} \right\}. \end{aligned}$$

This shows that

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\Gamma_j^R), C(\Gamma_k^L)) \\ & \cong C(\emptyset) \left\{ \begin{bmatrix} n+j+k-m \\ k \end{bmatrix} \begin{bmatrix} n+j+k-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-k \end{bmatrix} \begin{bmatrix} N+m-n-j-k \\ m-j \end{bmatrix} \begin{bmatrix} N \\ n+j+k-m \end{bmatrix} q^{(m+n)N-n^2-m^2} \right\}. \end{aligned}$$

The rest of the lemma follows from the above isomorphism. \square

Corollary 11.12. *Let m, n be integers such that $0 \leq m, n \leq N$. For $\max\{m-n, 0\} \leq k \leq m$, the matrix factorizations $C(\Gamma_k^L)$ and $C(\Gamma_k^R)$ are naturally homotopic in the sense that the homotopy equivalence $C(\Gamma_k^L) \xrightarrow{\sim} C(\Gamma_k^R)$ and $C(\Gamma_k^R) \xrightarrow{\sim} C(\Gamma_k^L)$ are unique up to homotopy and scaling.*

Proof. The existence of the homotopy equivalence is from Lemma 11.10. The uniqueness follows from the $j = k$ case of Lemma 11.11. \square

Corollary 11.13. *Let m, n be integers such that $0 \leq m, n \leq N$. For $\max\{m - n, 0\} \leq j, k \leq m$, up to homotopy and scaling, there exist unique homogeneous morphisms*

$$\begin{aligned} d_{j,k}^{LL} &: C(\Gamma_j^L) \rightarrow C(\Gamma_k^L), \\ d_{j,k}^{RL} &: C(\Gamma_j^R) \rightarrow C(\Gamma_k^L), \\ d_{j,k}^{LR} &: C(\Gamma_j^L) \rightarrow C(\Gamma_k^R), \\ d_{j,k}^{RR} &: C(\Gamma_j^R) \rightarrow C(\Gamma_k^R) \end{aligned}$$

satisfying

- $d_{j,k}^{LL}, d_{j,k}^{RL}, d_{j,k}^{LR}$ and $d_{j,k}^{RR}$ have quantum degree $(j - k)^2$ and \mathbb{Z}_2 -degree 0,
- $d_{j,k}^{LL}, d_{j,k}^{RL}, d_{j,k}^{LR}$ and $d_{j,k}^{RR}$ are homotopically non-trivial.

Moreover, up to homotopy and scaling, every square in the following diagram commutes, where the vertical morphisms are either identity or the natural homotopy equivalence.

$$\begin{array}{ccc} C(\Gamma_j^L) & \xrightarrow{d_{j,k}^{LL}} & C(\Gamma_k^L) \\ \downarrow \simeq & & \downarrow = \\ C(\Gamma_j^R) & \xrightarrow{d_{j,k}^{RL}} & C(\Gamma_k^L) \\ \downarrow \simeq & & \downarrow \simeq \\ C(\Gamma_j^L) & \xrightarrow{d_{j,k}^{LR}} & C(\Gamma_k^R) \\ \downarrow \simeq & & \downarrow = \\ C(\Gamma_j^R) & \xrightarrow{d_{j,k}^{RR}} & C(\Gamma_k^R) \end{array}$$

Proof. This corollary follows easily from Lemma 11.11. \square

From Corollary 11.13, we know that, up to homotopy and scaling, the morphisms $d_{j,k}^{LL}, d_{j,k}^{RL}, d_{j,k}^{LR}$ and $d_{j,k}^{RR}$ are identified with each other under the natural homotopy equivalence $C(\Gamma_j^L) \simeq C(\Gamma_j^R)$ and $C(\Gamma_k^L) \simeq C(\Gamma_k^R)$. So, without creating any confusion, we drop the superscripts in the notations and simply denote these morphisms by $d_{j,k}$.

Definition 11.14. Let m, n be integers such that $0 \leq m, n \leq N$. For $\max\{m - n, 0\} + 1 \leq k \leq m$, define $d_k^+ = d_{k,k-1}$. For $\max\{m - n, 0\} \leq k \leq m - 1$, define $d_k^- = d_{k,k+1}$. Note that these are homogeneous morphisms of quantum degree 1 and \mathbb{Z}_2 -degree 0.

Theorem 11.15. *Let m, n be integers such that $0 \leq m, n \leq N$. For $\max\{m - n, 0\} + 2 \leq k \leq m$, $d_{k-1}^+ \circ d_k^+ \simeq 0$. For $\max\{m - n, 0\} \leq k \leq m - 2$, $d_{k+1}^- \circ d_k^- \simeq 0$.*

Proof. For $\max\{m - n, 0\} + 2 \leq k \leq m$, $d_{k-1}^+ \circ d_k^+ : C(\Gamma_k^L) \rightarrow C(\Gamma_{k-2}^L)$ is a homogeneous morphism of quantum degree 2. But, by Lemma 11.11, the lowest

non-vanishing quantum grading of $\text{Hom}_{\text{HMF}}(C(\Gamma_k^L), C(\Gamma_{k-2}^L))$ is $2^2 = 4$. This implies that $d_{k-1}^+ \circ d_k^+ \simeq 0$. The proof of $d_{k+1}^- \circ d_k^- \simeq 0$ is very similar and left to the reader. \square

Definition 11.16. Let $c_{m,n}^\pm$ be the colored crossings in Figure 59, $\hat{R} = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$ and

$$w = p_{N+1}(\mathbb{X}) + p_{N+1}(\mathbb{Y}) - p_{N+1}(\mathbb{A}) - p_{N+1}(\mathbb{B}).$$

We first define the unnormalized chain complexes $\hat{C}(c_{m,n}^\pm)$.

If $m \leq n$, then $\hat{C}(c_{m,n}^+)$ is defined to be the object

$$0 \rightarrow C(\Gamma_m^L) \xrightarrow{d_m^+} C(\Gamma_{m-1}^L)\{q^{-1}\} \xrightarrow{d_{m-1}^+} \cdots \xrightarrow{d_1^+} C(\Gamma_0^L)\{q^{-m}\} \rightarrow 0$$

of $\text{hCh}^b(\text{hmf}_{\hat{R},w})$, where the homological grading on $\hat{C}(c_{m,n}^+)$ is defined so that the term $C(\Gamma_k^L)\{q^{-(m-k)}\}$ have homological grading $m - k$.

If $m > n$, then $\hat{C}(c_{m,n}^+)$ is defined to be the object

$$0 \rightarrow C(\Gamma_m^L) \xrightarrow{d_m^+} C(\Gamma_{m-1}^L)\{q^{-1}\} \xrightarrow{d_{m-1}^+} \cdots \xrightarrow{d_{m-n+1}^+} C(\Gamma_{m-n}^L)\{q^{-n}\} \rightarrow 0$$

of $\text{hCh}^b(\text{hmf}_{\hat{R},w})$, where the homological grading on $\hat{C}(c_{m,n}^+)$ is defined so that the term $C(\Gamma_k^L)\{q^{-(m-k)}\}$ has homological grading $m - k$.

If $m \leq n$, then $\hat{C}(c_{m,n}^-)$ is defined to be the object

$$0 \rightarrow C(\Gamma_0^L)\{q^m\} \xrightarrow{d_0^-} \cdots \xrightarrow{d_{m-2}^-} C(\Gamma_{m-1}^L)\{q\} \xrightarrow{d_{m-1}^-} C(\Gamma_m^L) \rightarrow 0$$

of $\text{hCh}^b(\text{hmf}_{\hat{R},w})$, where the homological grading on $\hat{C}(c_{m,n}^-)$ is defined so that the term $C(\Gamma_k^L)\{q^{m-k}\}$ has homological grading $k - m$.

If $m > n$, then $\hat{C}(c_{m,n}^-)$ is defined to be the object

$$0 \rightarrow C(\Gamma_{m-n}^L)\{q^n\} \xrightarrow{d_{m-n}^-} \cdots \xrightarrow{d_{m-2}^-} C(\Gamma_{m-1}^L)\{q\} \xrightarrow{d_{m-1}^-} C(\Gamma_m^L) \rightarrow 0$$

of $\text{hCh}^b(\text{hmf}_{\hat{R},w})$, where the homological grading on $\hat{C}(c_{m,n}^-)$ is defined so that the term $C(\Gamma_k^L)\{q^{m-k}\}$ has homological grading $k - m$.

The normalized chain complex $C(c_{m,n}^\pm)$ is defined to be

$$\begin{aligned} C(c_{m,n}^+) &= \begin{cases} \hat{C}(c_{m,n}^+) \langle m \rangle \parallel -m \parallel \{q^{m(N+1-m)}\} & \text{if } m = n, \\ \hat{C}(c_{m,n}^+) & \text{if } m \neq n, \end{cases} \\ C(c_{m,n}^-) &= \begin{cases} \hat{C}(c_{m,n}^-) \langle m \rangle \parallel m \parallel \{q^{-m(N+1-m)}\} & \text{if } m = n, \\ \hat{C}(c_{m,n}^-) & \text{if } m \neq n. \end{cases} \end{aligned}$$

(Recall that $\parallel m \parallel$ means shifting the homological grading by m . See Definition 2.33.)

Corollary 11.17. Replacing the left resolutions Γ_k^L in Definition 11.16 by right resolutions Γ_k^R does not change the isomorphism types of $\hat{C}(c_{m,n}^\pm)$ and $C(c_{m,n}^\pm)$ as objects of $\text{Ch}^b(\text{hmf}_{\hat{R},w})$.

Proof. This is an easy consequence of Lemma 11.10 and Corollaries 11.12 and 11.13. \square

Corollary 11.18. *The isomorphism type of the chain complexes $\hat{C}(D)$ and $C(D)$ associated to a knotted MOY graph D (see Definition 11.4) is independent of the choice of the marking of D .*

Proof. We only need to show that adding or removing an extra marked point on a segment of D does not change the isomorphism type. Note that adding or removing such an extra marked point is equivalent to adding or removing an internal marked point in a piece of D next to this extra marked point.

If the adjacent piece is of type (i) or (ii), i.e. an (embedded) MOY graph, then, by Lemma 5.6, adding or removing an internal marked point does not change the homotopy type of the matrix factorization of this piece. Moreover, it is easy to see that the chain map of this piece is 0 with or without the extra internal marked point. So, in this case, the addition or removal of the extra marked point does not change the isomorphism type of $C(D)$.

If the adjacent piece is of type (iii), i.e. a colored crossing, then, by Lemma 5.6, adding or removing an internal marked point does not change the homotopy types of the matrix factorizations associated to the resolutions of this colored crossing. Moreover, by the uniqueness part of Corollary 11.13, up to homotopy and scaling, the differential map is the same with or without the extra internal marked point. So, again, the addition or removal of the extra marked point does not change the isomorphism type of $C(D)$. \square

11.4. A null-homotopic chain complex. In this subsection, we construct a null-homotopic chain complex that will be useful in our proof of the invariance under fork sliding. The construction of this chain complex is similar to the chain complex of a colored crossing.

The next lemma is a special case of Decomposition (V) (Theorem 10.1.)

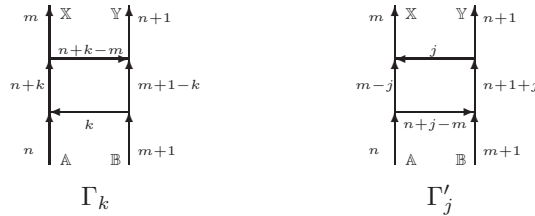


FIGURE 62.

Lemma 11.19. *Let m, n be integers such that $0 \leq m, n \leq N-1$. For $\max\{m-n, 0\} \leq k \leq m+1$ and $\max\{m-n, 0\} \leq j \leq m$, define Γ_k and Γ'_j to be the MOY graphs in Figure 62. Then, for $\max\{m-n, 0\} \leq k \leq m+1$,*

$$C(\Gamma_k) \simeq \begin{cases} C(\Gamma'_m) & \text{if } k = m+1, \\ C(\Gamma'_k) \oplus C(\Gamma'_{k-1}) & \text{if } \max\{m-n, 0\} + 1 \leq k \leq m, \\ C(\Gamma'_{\max\{m-n, 0\}}) & \text{if } k = \max\{m-n, 0\}. \end{cases}$$

Lemma 11.20. *Let Γ_k and Γ'_j be as in Lemma 11.19. Then*

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\Gamma'_j), C(\Gamma_k)) \\ & \cong C(\emptyset) \left\{ \begin{bmatrix} n+k+j-m \\ k \end{bmatrix} \begin{bmatrix} n+k+j-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m-j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m+1-k \end{bmatrix} \begin{bmatrix} N \\ n+k+j-m \end{bmatrix} q^{(m+n+1)(N-1)-m^2-n^2} \right\}. \end{aligned}$$

In particular, the space is concentrated on \mathbb{Z}_2 -grading 0. The lowest non-vanishing quantum grading of the above space is $(j-k)(j-k+1)$. Moreover, the subspace of homogeneous elements of quantum degree $(j-k)(j-k+1)$ is 1-dimensional.

Proof. By Lemma 11.9, we have

$$\mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma'_j), C(\Gamma_k)) \cong H(C(\Gamma_k) \otimes_{\hat{R}} C(\overline{\Gamma'_j})) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2}\},$$

where $\hat{R} = \mathrm{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$ and $\overline{\Gamma'_j}$ is Γ'_j with the opposite orientation.

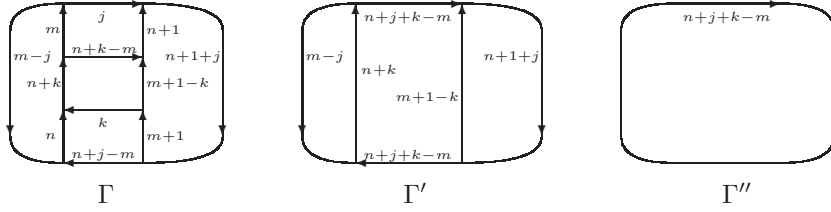


FIGURE 63.

Let Γ , Γ' and Γ'' be the MOY graphs in Figure 63. Then, by Corollary 5.13 and Decompositions (I-II) (Theorems 5.16 and 5.14), we have

$$\begin{aligned} & C(\Gamma_k) \otimes_{\hat{R}} C(\overline{\Gamma'_j}) = C(\Gamma) \\ & \simeq C(\Gamma') \left\{ \begin{bmatrix} n+k+j-m \\ k \end{bmatrix} \begin{bmatrix} n+k+j-m \\ j \end{bmatrix} \right\} \\ & \simeq C(\Gamma'') \langle j+k+1 \rangle \left\{ \begin{bmatrix} n+k+j-m \\ k \end{bmatrix} \begin{bmatrix} n+k+j-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m-j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m+1-k \end{bmatrix} \right\} \\ & \simeq C(\emptyset) \langle m+n+1 \rangle \left\{ \begin{bmatrix} n+k+j-m \\ k \end{bmatrix} \begin{bmatrix} n+k+j-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m-j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m+1-k \end{bmatrix} \begin{bmatrix} N \\ n+k+j-m \end{bmatrix} \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{HMF}}(C(\Gamma'_j), C(\Gamma_k)) \\ & \simeq C(\emptyset) \left\{ \begin{bmatrix} n+k+j-m \\ k \end{bmatrix} \begin{bmatrix} n+k+j-m \\ j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m-j \end{bmatrix} \begin{bmatrix} N+m-n-k-j \\ m+1-k \end{bmatrix} \begin{bmatrix} N \\ n+k+j-m \end{bmatrix} q^{(m+n+1)(N-1)-m^2-n^2} \right\}. \end{aligned}$$

The rest of the lemma follows from this isomorphism. \square

Lemma 11.21. For $\max\{m-n, 0\} \leq i, j \leq m$,

$$\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_j)) \cong \begin{cases} \mathbb{C} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In the case $i = j$, $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_i))$ is spanned by $\mathrm{id}_{C(\Gamma'_i)}$.

Proof. If $i > j$, then by Lemma 11.20, $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_j)) = 0$. But, by Lemma 11.19, $C(\Gamma_j) = C(\Gamma'_j) \oplus C(\Gamma'_{j-1})$. This implies that $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_j)) \cong 0$.

If $i < j$, then by Lemma 11.20, $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_{j+1})) = 0$. But, by Lemma 11.19, $C(\Gamma_{j+1}) = C(\Gamma'_{j+1}) \oplus C(\Gamma'_j)$. This implies that $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_j)) \cong 0$.

If $i = j$, then by Lemma 11.20, $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_{i+1})) \cong \mathbb{C}$. But, by Lemma 11.19, $C(\Gamma_{i+1}) = C(\Gamma'_{i+1}) \oplus C(\Gamma'_i)$ and, from above, $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_i), C(\Gamma'_{i+1})) = 0$.

This implies that $\text{Hom}_{\text{hmf}}(C(\Gamma'_i), C(\Gamma'_i)) \cong \mathbb{C}$. It follows that $C(\Gamma'_i)$ is not null-homotopic and, therefore, $\text{id}_{C(\Gamma'_i)}$ is not null-homotopic. So $\text{id}_{C(\Gamma'_i)}$ spans the 1-dimensional space $\text{Hom}_{\text{hmf}}(C(\Gamma'_i), C(\Gamma'_i))$. \square

Lemma 11.22. *For $\max\{m - n, 0\} \leq j, k \leq m + 1$,*

$$\text{Hom}_{\text{hmf}}(C(\Gamma_j), C(\Gamma_k)) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C} & \text{if } \max\{m - n, 0\} + 1 \leq j = k \leq m, \\ \mathbb{C} & \text{if } j = k = \max\{m - n, 0\} \text{ or } m + 1, \\ \mathbb{C} & \text{if } |j - k| = 1, \\ 0 & \text{if } |j - k| > 1. \end{cases}$$

Proof. This follows easily from Lemmas 11.19 and 11.21. \square

Definition 11.23. Denote by

$$\begin{aligned} J_{k,k} &: C(\Gamma'_k) \rightarrow C(\Gamma_k) \\ J_{k,k-1} &: C(\Gamma'_{k-1}) \rightarrow C(\Gamma_k) \\ P_{k,k} &: C(\Gamma_k) \rightarrow C(\Gamma'_k) \\ P_{k,k-1} &: C(\Gamma_k) \rightarrow C(\Gamma'_{k-1}) \end{aligned}$$

the inclusion and projection morphisms in the decomposition

$$C(\Gamma_k) \simeq C(\Gamma'_k) \oplus C(\Gamma'_{k-1}).$$

Define

$$\begin{aligned} \delta_k^+ &= J_{k-1,k-1} \circ P_{k,k-1} : C(\Gamma_k) \rightarrow C(\Gamma_{k-1}), \\ \delta_k^- &= J_{k+1,k} \circ P_{k,k} : C(\Gamma_k) \rightarrow C(\Gamma_{k+1}). \end{aligned}$$

δ_k^+ and δ_k^- are both homotopically non-trivial homogeneous morphisms preserving both the \mathbb{Z}_2 -grading and the quantum grading. By Lemma 11.22, up to homotopy and scaling, δ_k^+ and δ_k^- are the unique morphisms with such properties.

Lemma 11.24. $\delta_{k-1}^+ \circ \delta_k^+ \simeq 0$, $\delta_{k+1}^- \circ \delta_k^- \simeq 0$.

Proof. From Lemma 11.22, we have that

$$\text{Hom}_{\text{hmf}}(C(\Gamma_k), C(\Gamma_{k-2})) \cong \text{Hom}_{\text{hmf}}(C(\Gamma_k), C(\Gamma_{k+2})) \cong 0.$$

The lemma follows from this. \square

Let $\hat{R} = \text{Sym}(\mathbb{X}|\mathbb{Y}|\mathbb{A}|\mathbb{B})$ and $w = p_{N+1}(\mathbb{X}) + p_{N+1}(\mathbb{Y}) - p_{N+1}(\mathbb{A}) - p_{N+1}(\mathbb{B})$. The above discussion implies the following.

Proposition 11.25. *Let k_1 and k_2 be integers such that $\max\{m - n, 0\} + 1 \leq k_1 \leq k_2 \leq m$. Then*

$$\begin{aligned} 0 \longrightarrow C(\Gamma'_{k_2}) &\xrightarrow{J_{k_2,k_2}} C(\Gamma_{k_2}) \xrightarrow{\delta_{k_2}^+} \cdots \xrightarrow{\delta_{k_1+1}^+} C(\Gamma_{k_1}) \xrightarrow{P_{k_1,k_1-1}} C(\Gamma'_{k_1-1}) \longrightarrow 0, \\ 0 \longrightarrow C(\Gamma'_{k_1-1}) &\xrightarrow{J_{k_1,k_1-1}} C(\Gamma_{k_1}) \xrightarrow{\delta_{k_1}^-} \cdots \xrightarrow{\delta_{k_2-1}^-} C(\Gamma_{k_2}) \xrightarrow{P_{k_2,k_2}} C(\Gamma'_{k_2}) \longrightarrow 0 \end{aligned}$$

are both chain complexes over $\text{hmf}_{\hat{R},w}$ and are isomorphic in $\text{Ch}^b(\text{hmf}_{\hat{R},w})$ to

$$\bigoplus_{j=k_1-1}^{k_2} (0 \longrightarrow C(\Gamma'_j) \xrightarrow{\simeq} C(\Gamma'_j) \longrightarrow 0,)$$

which is homotopic to 0 (i.e. isomorphic in $\text{hCh}^b(\text{hmf}_{\hat{R},w})$ to 0.)

11.5. Explicit forms of the differential maps. In the proof of the invariance of the $\mathfrak{sl}(N)$ -homology, we need to use explicit forms of the differential maps in the chain complexes defined in the previous two subsections. In this subsection, we give one construction of such explicit forms. (There are more than one explicit constructions of the same differential maps. See e.g. [27, Figure 17].)

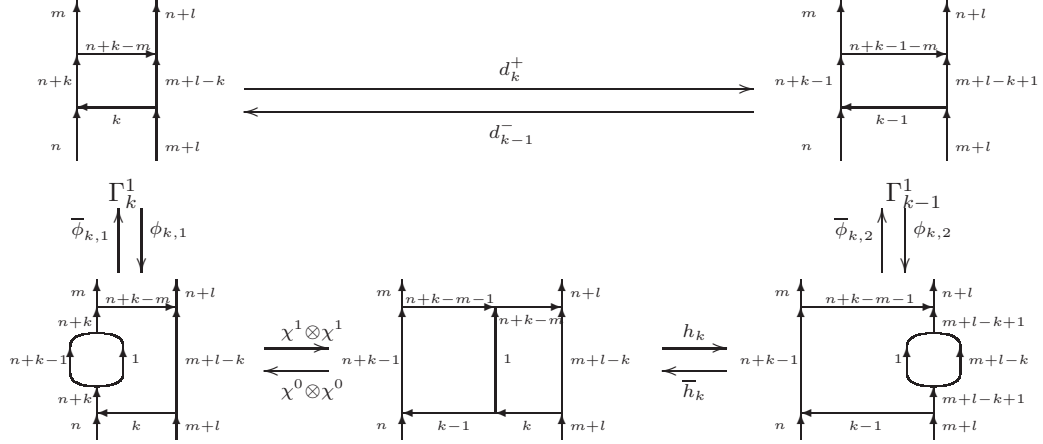


FIGURE 64.

Consider the MOY graphs and the morphisms in Figure 64. Here, $\phi_{k,1}$, $\bar{\phi}_{k,1}$, $\phi_{k,2}$, $\bar{\phi}_{k,2}$ are the morphisms associated to the apparent edge splittings and mergings define in Definition 7.10. χ^0 and χ^1 are the morphisms from Proposition 7.20 (more precisely, Corollary 9.3.) h_k , \bar{h}_k are the morphisms induced by the bouquet moves. (See Corollary 5.13, Lemma 7.4 and Remark 7.5.) d_k^+ and d_{k-1}^- are defined by

$$\begin{aligned} d_k^+ &= \bar{\phi}_{k,2} \circ h_k \circ (\chi^1 \otimes \chi^1) \circ \phi_{k,1}. \\ d_{k-1}^- &= \bar{\phi}_{k,1} \circ (\chi^0 \otimes \chi^0) \circ \bar{h}_k \circ \phi_{k,2}. \end{aligned}$$

Theorem 11.26. d_k^+ and d_{k-1}^- are homotopically non-trivial homogeneous morphisms of \mathbb{Z}_2 -degree 0 and quantum degree $1-l$.

When $l=0$, d_k^+ and d_{k-1}^- are explicit forms of the differential maps of the chain complexes associated to colored crossings defined in Definition 11.16.

When $l=1$, d_k^+ and d_{k-1}^- are explicit forms of the differential maps δ_k^+ and δ_{k-1}^- of the null-homotopic chain complexes in Proposition 11.25.

Consider the diagram in Figure 65, where the morphisms are induced by the apparent local changes of MOY graphs. We have the following lemma.

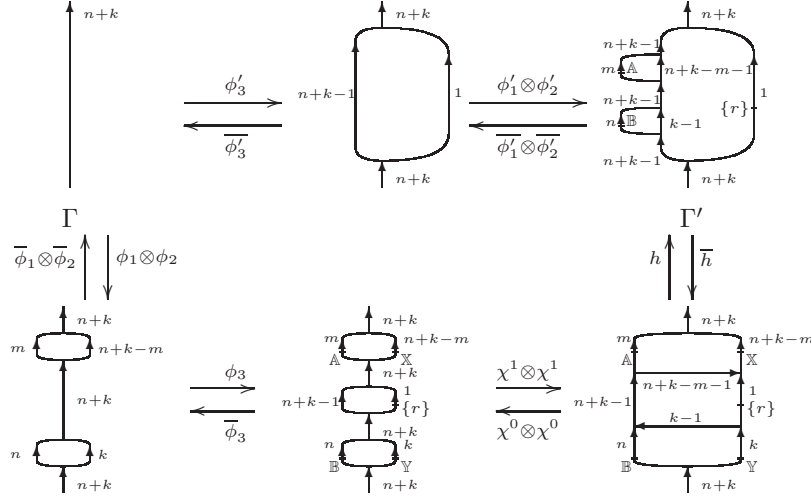


FIGURE 65.

Lemma 11.27.

$$\begin{aligned} (\phi'_1 \otimes \phi'_2) \circ \phi'_3 &\approx h \circ (\chi^1 \otimes \chi^1) \circ \phi_3 \circ (\phi_1 \otimes \phi_2), \\ \overline{\phi'_3} \circ (\overline{\phi'_1} \otimes \overline{\phi'_2}) &\approx (\overline{\phi_1} \otimes \overline{\phi_2}) \circ \overline{\phi_3} \circ (\chi^0 \otimes \chi^0) \circ \overline{h}. \end{aligned}$$

That is, the diagram in Figure 65 commutes up to homotopy and scaling in both directions.

Proof. Let

$$\begin{aligned} f &= (\phi'_1 \otimes \phi'_2) \circ \phi'_3, & \overline{f} &= \overline{\phi'_3} \circ (\overline{\phi'_1} \otimes \overline{\phi'_2}), \\ g &= h \circ (\chi^1 \otimes \chi^1) \circ \phi_3 \circ (\phi_1 \otimes \phi_2), & \overline{g} &= (\overline{\phi_1} \otimes \overline{\phi_2}) \circ \overline{\phi_3} \circ (\chi^0 \otimes \chi^0) \circ \overline{h}. \end{aligned}$$

Then $f, \overline{f}, g, \overline{g}$ are homogeneous morphisms of \mathbb{Z}_2 -degree 0 and quantum degree $\tau := m - k + 1 - m(n + k - m) - nk$.

Using Decomposition II (Theorem 5.14), we have

$$C(\Gamma') \simeq C(\Gamma) \{ [n+k] \begin{bmatrix} n+k-1 \\ m \end{bmatrix} \begin{bmatrix} n+k-1 \\ n \end{bmatrix} \}.$$

Denote by \bigcirc_{n+k} an oriented circle colored by $n+k$. It is easy to check that

$$\begin{aligned} \text{Hom}_{\text{HMF}}(C(\Gamma), C(\Gamma')) &\cong \text{Hom}_{\text{HMF}}(C(\Gamma'), C(\Gamma)) \\ &\cong H(\bigcirc_{n+k}) \langle n+k \rangle \{ [n+k] \begin{bmatrix} n+k-1 \\ m \end{bmatrix} \begin{bmatrix} n+k-1 \\ n \end{bmatrix} q^{(n+k)(N-n-k)} \} \\ &\cong C(\emptyset) \{ \begin{bmatrix} N \\ n+k \end{bmatrix} [n+k] \begin{bmatrix} n+k-1 \\ m \end{bmatrix} \begin{bmatrix} n+k-1 \\ n \end{bmatrix} q^{(n+k)(N-n-k)} \}. \end{aligned}$$

These spaces are concentrated on \mathbb{Z}_2 -grading 0. The lowest non-vanishing quantum grading of these spaces is τ , and the subspaces of these spaces of homogeneous elements of quantum grading τ are 1-dimensional. Thus, to show that $f \approx g$ and $\overline{f} \approx \overline{g}$, we only need to show that $f, \overline{f}, g, \overline{g}$ are all homotopically non-trivial.

By Lemma 7.11, we have

$$\overline{f} \circ \mathfrak{m}(S_{\lambda_{m, n+k-1-m}}(\mathbb{A}) \cdot S_{\lambda_{n, k-1}}(\mathbb{B}) \cdot (-r)^{n+k-1}) \circ f \approx \text{id}_{C(\Gamma)}.$$

This implies that f, \bar{f} are not homotopic to 0.

By Corollary 9.3, we have

$$\begin{aligned}
& \bar{g} \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(-\mathbb{X}) \cdot S_{\lambda_{n,k-1}}(-\mathbb{Y}) \cdot (-r)^{n+k-1}) \circ g \\
& \approx (\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \bar{\phi}_3 \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(-\mathbb{X}) \cdot S_{\lambda_{n,k-1}}(-\mathbb{Y}) \cdot (-r)^{n+k-1}) \circ (\chi^0 \otimes \chi^0) \circ (\chi^1 \otimes \chi^1) \circ \phi_3 \circ (\phi_1 \otimes \phi_2) \\
& \approx (\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \bar{\phi}_3 \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(-\mathbb{X}) \cdot S_{\lambda_{n,k-1}}(-\mathbb{Y}) \cdot (-r)^{n+k-1} \cdot (\sum_{j=0}^m (-r)^{m-j} A_j) \cdot (\sum_{i=0}^n (-r)^{n-i} B_i)) \circ \phi_3 \circ (\phi_1 \otimes \phi_2) \\
& = \sum_{j=0}^m \sum_{i=0}^n (\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \bar{\phi}_3 \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(-\mathbb{X}) \cdot A_j \cdot S_{\lambda_{n,k-1}}(-\mathbb{Y}) \cdot B_i \cdot (-r)^{2n+m+k-1-i-j}) \circ \phi_3 \circ (\phi_1 \otimes \phi_2),
\end{aligned}$$

where A_j, B_j are the j -th elementary symmetric polynomials in \mathbb{A} and \mathbb{B} . But, by Lemma 7.11, the only homotopically non-trivial term on the right hand side is the one with $j = m, i = n$. So

$$\bar{g} \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(-\mathbb{X}) \cdot S_{\lambda_{n,k-1}}(-\mathbb{Y}) \cdot (-r)^{n+k-1}) \circ g \approx \bar{\phi}_3 \circ \mathfrak{m}((-r)^{n+k-1}) \circ \phi_3 \approx \text{id}_{C(\Gamma)}.$$

Thus, g, \bar{g} are not homotopic to 0. \square

Proof of Theorem 11.26. It is easy to check that d_k^+ and d_{k-1}^- are homogeneous morphisms of \mathbb{Z}_2 -degree 0 and quantum degree $1-l$. Recall that the differential maps of the complexes in Definition 11.16 and Proposition 11.25 are homotopically non-trivial homogeneous morphisms uniquely determined up to homotopy and scaling by their quantum degrees. So, to prove Theorem 11.26, we only need to prove that d_k^+ and d_{k-1}^- are not null-homotopic.

Consider the MOY graphs in Figure 66, where the morphisms are induced by the apparent local changes of the MOY graphs. Note that, as morphisms between $C(\hat{\Gamma}_k)$ and $C(\hat{\Gamma}_{k-1})$,

$$\begin{aligned}
d_k^+ &= \bar{\phi}_4 \circ h_2 \circ (\chi^1 \otimes \chi^1) \circ \phi_3, \\
d_{k-1}^- &= \bar{\phi}_3 \circ (\chi^0 \otimes \chi^0) \circ \bar{h}_2 \circ \phi_4.
\end{aligned}$$

So, by Lemma 11.27, we have

$$\begin{aligned}
h_3 \circ d_k^+ \circ h_1 \circ (\phi_1 \otimes \phi_2) &\approx \bar{\phi}_4 \circ (\phi_5 \otimes \phi_6) \circ h_4 \circ \phi_3, \\
(\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \bar{h}_1 \circ d_{k-1}^- \bar{h}_3 &\approx \bar{\phi}_3 \circ \bar{h}_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ \phi_4 \approx \bar{\phi}_3 \circ \bar{h}_4 \circ \phi_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6),
\end{aligned}$$

where the morphisms on the right hand side are depicted in Figure 67. Note that some morphisms in Figures 66 and 67 are given the same notations. This is because they are induced by the same local changes of MOY graphs.

Note that $\bar{\phi}_7 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ \bar{\phi}_4 \approx \bar{\phi}_0 \circ \bar{\phi}_3 \circ \bar{h}_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6)$. So, by Lemma 7.11, we have

$$\begin{aligned}
& \bar{\phi}_7 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E}) \cdot B_{n+k-1}) \circ h_3 \circ d_k^+ \circ h_1 \circ (\phi_1 \otimes \phi_2) \circ \mathfrak{m}(S_{\lambda_{n+k,m+l-k}}(-\mathbb{X})) \circ \phi_0 \\
& \approx \bar{\phi}_7 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E}) \cdot B_{n+k-1}) \circ \bar{\phi}_4 \circ (\phi_5 \otimes \phi_6) \circ h_4 \circ \phi_3 \circ \mathfrak{m}(S_{\lambda_{n+k,m+l-k}}(-\mathbb{X})) \circ \phi_0 \\
& \approx \bar{\phi}_7 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ \bar{\phi}_4 \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E}) \cdot B_{n+k-1}) \circ (\phi_5 \otimes \phi_6) \circ h_4 \circ \phi_3 \circ \mathfrak{m}(S_{\lambda_{n+k,m+l-k}}(-\mathbb{X})) \circ \phi_0 \\
& \approx \bar{\phi}_0 \circ \bar{\phi}_3 \circ \bar{h}_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ \mathfrak{m}(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E}) \cdot B_{n+k-1}) \circ (\phi_5 \otimes \phi_6) \circ h_4 \circ \phi_3 \circ \mathfrak{m}(S_{\lambda_{n+k,m+l-k}}(-\mathbb{X})) \circ \phi_0 \\
& \approx \bar{\phi}_0 \circ \mathfrak{m}(S_{\lambda_{n+k,m+l-k}}(-\mathbb{X})) \circ \phi_0 \approx \text{id}_{C(\Gamma)},
\end{aligned}$$

where B_{n+k-1} is the $(n+k-1)$ -th elementary symmetric polynomial in \mathbb{B} . This implies that d_k^+ is not null-homotopic.

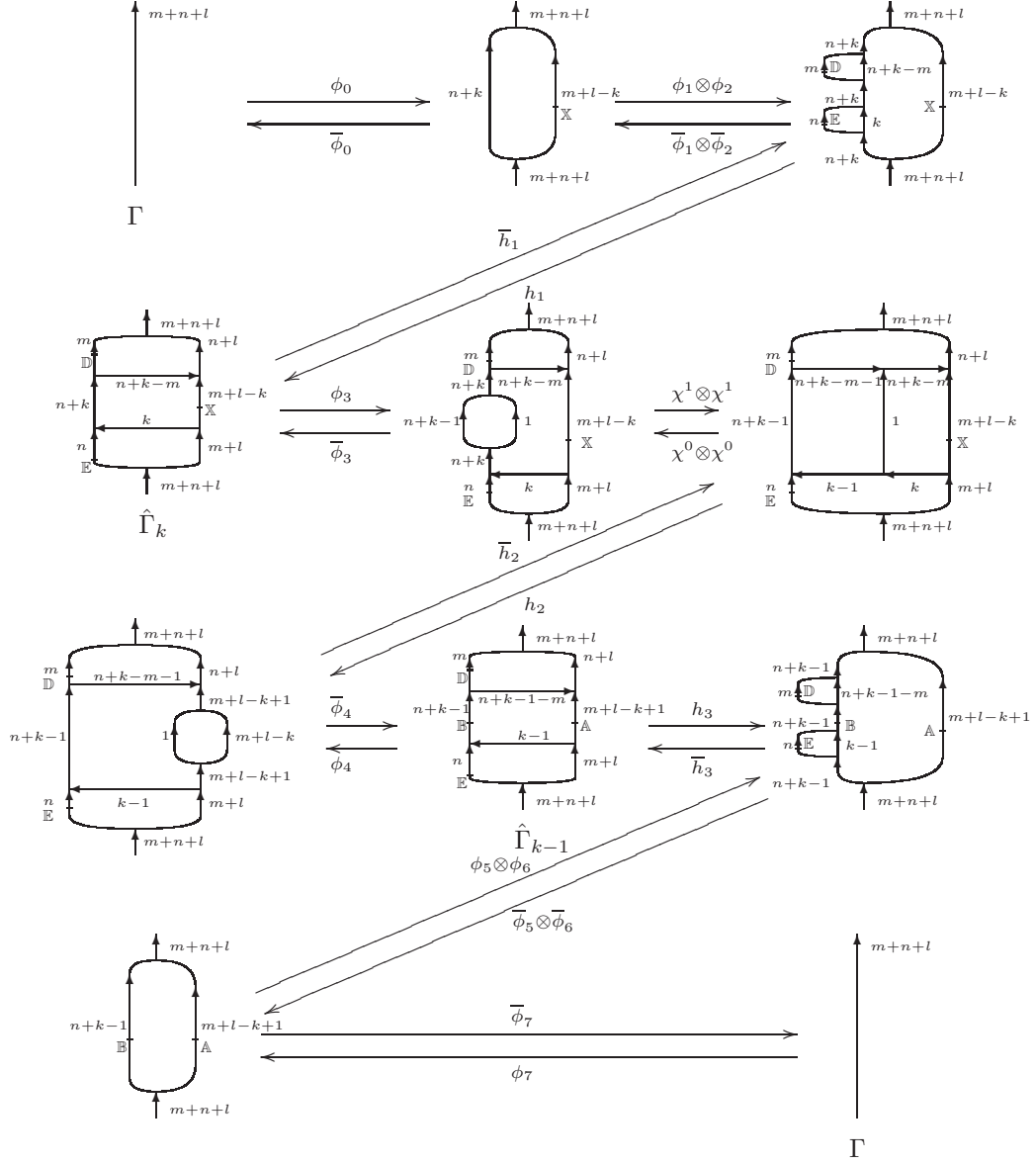


FIGURE 66.

Similarly, note that $\bar{\phi}_0 \circ \bar{\phi}_3 \approx \bar{\phi}_7 \circ \bar{\phi}_4 \circ h_4$. So, by Lemma 7.11, we have

$$\begin{aligned}
& \bar{\phi}_0 \circ m(X_{m+l-k}) \circ (\bar{\phi}_1 \otimes \bar{\phi}_2) \circ \bar{h}_1 \circ d_{k-1}^- \bar{h}_3 \circ m(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E})) \circ (\phi_5 \otimes \phi_6) \circ m(S_{\lambda_{n+k-1,m+l+1-k}}(\mathbb{B})) \circ \phi_7 \\
& \approx \bar{\phi}_0 \circ m(X_{m+l-k}) \circ \bar{\phi}_3 \circ \bar{h}_4 \circ \phi_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ m(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E})) \circ (\phi_5 \otimes \phi_6) \circ m(S_{\lambda_{n+k-1,m+l+1-k}}(\mathbb{B})) \circ \phi_7 \\
& \approx \bar{\phi}_0 \circ \bar{\phi}_3 \circ \bar{h}_4 \circ m(X_{m+l-k}) \circ \phi_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ m(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E})) \circ (\phi_5 \otimes \phi_6) \circ m(S_{\lambda_{n+k-1,m+l+1-k}}(\mathbb{B})) \circ \phi_7 \\
& \approx \bar{\phi}_7 \circ \bar{\phi}_4 \circ m(X_{m+l-k}) \circ \phi_4 \circ (\bar{\phi}_5 \otimes \bar{\phi}_6) \circ m(S_{\lambda_{m,n+k-1-m}}(\mathbb{D}) \cdot S_{\lambda_{n,k-1}}(\mathbb{E})) \circ (\phi_5 \otimes \phi_6) \circ m(S_{\lambda_{n+k-1,m+l+1-k}}(\mathbb{B})) \circ \phi_7 \\
& \approx \bar{\phi}_7 \circ m(S_{\lambda_{n+k-1,m+l+1-k}}(\mathbb{B})) \circ \phi_7 \approx \text{id}_C(\Gamma),
\end{aligned}$$

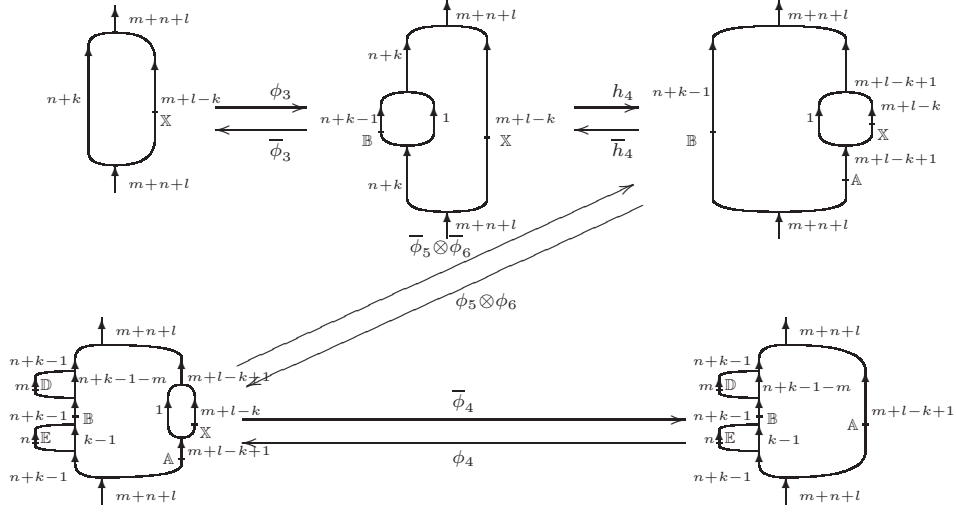


FIGURE 67.

where X_j is the j -th elementary symmetric polynomial in \mathbb{X} . This implies that d_{k-1}^- is not null-homotopic. \square

If, in a colored crossing, one of the two branches is colored by 1, then we have a simpler explicit description for the chain complex associated to this crossing.



FIGURE 68.

Consider the colored crossings $c_{1,n}^+$ and $c_{1,n}^-$ in Figure 68. Their MOY resolutions are given in Figure 69.

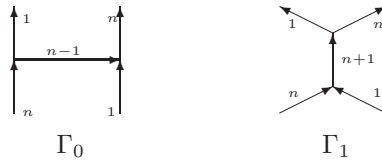


FIGURE 69.

Recall that, by Proposition 7.20 (or, more precisely, Corollary 8.9), there are homogeneous morphisms $\chi^0 : C(\Gamma_0) \rightarrow C(\Gamma_1)$ and $\chi^1 : C(\Gamma_1) \rightarrow C(\Gamma_0)$ that have \mathbb{Z}_2 -degree 0, quantum degree 1. By Proposition 7.29, up to homotopy and scaling, χ^0 and χ^1 are the unique homotopically non-trivial homogeneous morphisms with such degrees. Thus, we have the following corollary.

Corollary 11.28. *The unnormalized chain complexes of $c_{1,n}^+$ and $c_{1,n}^-$ are*

$$\begin{aligned}\hat{C}(c_{1,n}^+) &= "0 \rightarrow \underbrace{C(\Gamma_1)}_0 \xrightarrow{\chi^1} \underbrace{C(\Gamma_0)\{q^{-1}\}}_1 \rightarrow 0", \\ \hat{C}(c_{1,n}^-) &= "0 \rightarrow \underbrace{C(\Gamma_0)\{q\}}_{-1} \xrightarrow{\chi^0} \underbrace{C(\Gamma_1)}_0 \rightarrow 0",\end{aligned}$$

where the numbers in the under-braces are the homological gradings.

The differential maps in the chain complexes of $c_{m,1}^\pm$ can also be similarly expressed as the corresponding χ^0 and χ^1 . The details are left to the reader.

Remark 11.29. Corollary 11.28 shows that, for $c_{1,n}^\pm$ and $c_{m,1}^\pm$, the chain complexes defined in Definition 11.16 specialize to the corresponding chain complexes defined in [44]. In particular, for $c_{1,1}^\pm$, the chain complexes defined in Definition 11.16 specialize to the corresponding complexes in [18]. So our construction is a generalization of the Khovanov-Rozansky homology.

12. INVARIANCE UNDER FORK SLIDING

In this section, we prove the invariance of the homotopy type of the unnormalized chain complex associated to a knotted MOY graph under fork sliding. This is the most complex part of the proof of the invariance of the $\mathfrak{sl}(N)$ -homology. Once we have the invariance under fork sliding, the proof of the invariance reduces to an easy induction based on the invariance of the uncolored Khovanov-Rozansky $\mathfrak{sl}(N)$ -homology. Theorem 12.1 is the main result of this section.

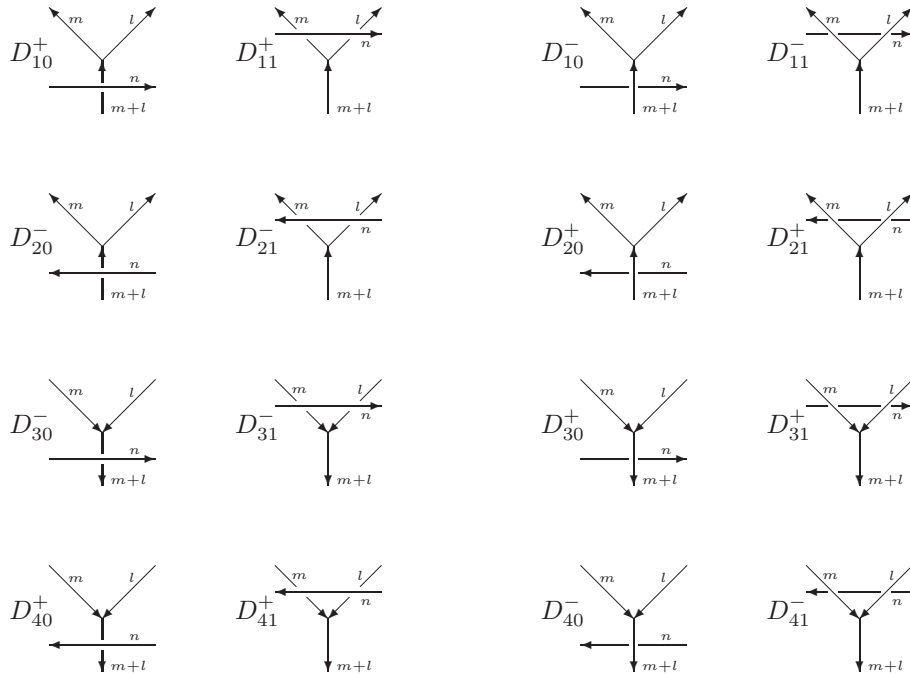


FIGURE 70.

Theorem 12.1. *Let $D_{i,j}^\pm$ be the knotted MOY graphs in Figure 70. Then $\hat{C}(D_{i,0}^+) \simeq \hat{C}(D_{i,1}^+)$ and $\hat{C}(D_{i,0}^-) \simeq \hat{C}(D_{i,1}^-)$. That is, $\hat{C}(D_{i,0}^+)$ (resp. $\hat{C}(D_{i,0}^-)$) is isomorphic in $\mathbf{hCh}^b(\mathbf{hmf})$ to $\hat{C}(D_{i,1}^+)$ (resp. $\hat{C}(D_{i,1}^-)$).*

We prove Theorem 12.1 by induction. The hardest part of the proof is to show that Theorem 12.1 is true for some special cases in which $m = 1$ or $l = 1$. Once we prove these special cases, the rest of the induction is quite easy. Next, we state these special cases of Theorem 12.1 separately as Proposition 12.2 and then use this proposition to prove Theorem 12.1. After that, we devote the rest of this section to prove Proposition 12.2.

Proposition 12.2. *Let $D_{i,j}^\pm$ be the knotted MOY graphs in Figure 70.*

- (i) *If $l = 1$, then $\hat{C}(D_{i,0}^+) \simeq \hat{C}(D_{i,1}^+)$ and $\hat{C}(D_{i,0}^-) \simeq \hat{C}(D_{i,1}^-)$ for $i = 1, 4$.*
- (ii) *If $m = 1$, then $\hat{C}(D_{i,0}^+) \simeq \hat{C}(D_{i,1}^+)$ and $\hat{C}(D_{i,0}^-) \simeq \hat{C}(D_{i,1}^-)$ for $i = 2, 3$.*

Proof of Theorem 12.1 (assuming Proposition 12.2 is true). Each homotopy equivalence in Theorem 12.1 can be proved by an induction on m or l . We only give details for the proof of

$$(12.1) \quad \hat{C}(D_{1,0}^+) \simeq \hat{C}(D_{1,1}^+).$$

The proof of the rest of Theorem 12.1 is very similar and left to the reader.

We prove (12.1) by an induction on l . The $l = 1$ case is covered by Part (i) of Proposition 12.2. Assume that (12.1) is true for some $l = k \geq 1$. Consider $l = k + 1$.

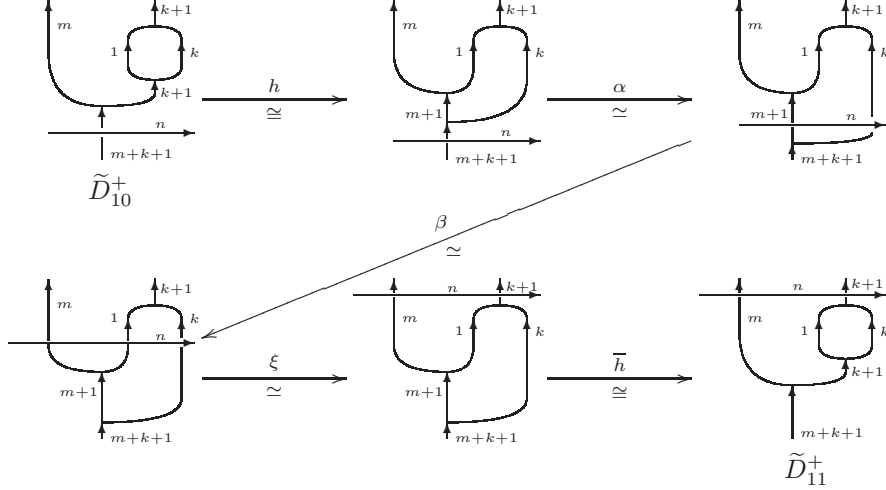


FIGURE 71.

Let \tilde{D}_{10}^+ and \tilde{D}_{11}^+ be the first and last knotted MOY graphs in Figure 71. By Decomposition (II) (Theorem 5.14), we have $\hat{C}(\tilde{D}_{10}^+) \cong \hat{C}(D_{10}^+)\{[k+1]\}$ and $\hat{C}(\tilde{D}_{11}^+) \cong \hat{C}(D_{11}^+)\{[k+1]\}$ in $\mathbf{Ch}^b(\mathbf{hmf})$. Consider the diagram in Figure 71. Here, h and \bar{h} are the isomorphisms in $\mathbf{Ch}^b(\mathbf{hmf})$ induced by the apparent bouquet moves. α is the isomorphisms in $\mathbf{hCh}^b(\mathbf{hmf})$ given by induction hypothesis. β is the isomorphisms in $\mathbf{hCh}^b(\mathbf{hmf})$ given by Part (i) of Proposition 12.2. ξ is also the isomorphisms in $\mathbf{hCh}^b(\mathbf{hmf})$ given by Part (i) of Proposition 12.2. Altogether, we

have

$$\hat{C}(D_{10}^+)[[k+1]] \cong \hat{C}(\tilde{D}_{10}^+) \simeq \hat{C}(\tilde{D}_{11}^+) \cong \hat{C}(D_{11}^+)[[k+1]].$$

So, by Proposition 3.20, $\hat{C}(D_{10}^+) \simeq \hat{C}(D_{11}^+)$ when $l = k+1$. \square

In the remainder of this section, we concentrate on proving Proposition 12.2. We only give the detailed proofs of $\hat{C}(D_{1,0}^\pm) \simeq \hat{C}(D_{1,1}^\pm)$ when $l = 1$. The proof of the rest of Proposition 12.2 is very similar and left to the reader.

12.1. Chain complexes involved in the proof. In the rest of this section, we fix $l = 1$. Then D_{10}^\pm and D_{11}^\pm are the knotted MOY graphs in Figure 72. Recall that we are trying to prove that

$$(12.2) \quad \hat{C}(D_{10}^\pm) \simeq \hat{C}(D_{11}^\pm) \text{ if } l = 1.$$

Several chain complexes appear in the proof of (12.2). We list these chain complexes in this subsection.



FIGURE 72.

Note that there is only one crossing in D_{10}^\pm , which is of the type $c_{m+1,n}^\pm$. We denote by \tilde{d}_k^\pm the differential map of $\hat{C}(c_{m+1,n}^\pm)$.

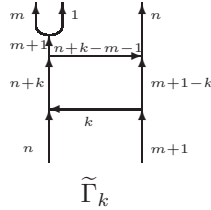


FIGURE 73.

Denote by $\tilde{\Gamma}_k$ the MOY graph in Figure 73. Then $\hat{C}(D_{10}^+)$ is

$$(12.3) \quad 0 \rightarrow C(\tilde{\Gamma}_{m+1}) \xrightarrow{\tilde{d}_{m+1}^+} C(\tilde{\Gamma}_m)\{q^{-1}\} \xrightarrow{\tilde{d}_m^+} \dots \xrightarrow{\tilde{d}_{\tilde{k}_0+1}^+} C(\tilde{\Gamma}_{\tilde{k}_0})\{q^{\tilde{k}_0-m-1}\} \rightarrow 0,$$

where $\tilde{k}_0 := \max\{0, m+1-n\}$. Similarly, $\hat{C}(D_{10}^-)$ is

$$(12.4) \quad 0 \rightarrow C(\tilde{\Gamma}_{\tilde{k}_0})\{q^{m+1-\tilde{k}_0}\} \xrightarrow{\tilde{d}_{\tilde{k}_0}^-} \dots \xrightarrow{\tilde{d}_{m-1}^-} C(\tilde{\Gamma}_m)\{q\} \xrightarrow{\tilde{d}_m^-} C(\tilde{\Gamma}_{m+1}) \rightarrow 0.$$

Let Γ'_k and Γ''_k be the MOY graphs in Figure 74. Let $\delta_k^\pm : C(\Gamma'_k) \rightarrow C(\Gamma'_{k \mp 1})$ be the morphisms defined in Definition 11.23 with explicit form given in Theorem 11.26. Let C^+ be the chain complex

$$(12.5) \quad 0 \rightarrow C(\Gamma''_{m-1}) \xrightarrow{J_{m-1,m-1}} C(\Gamma'_{m-1}) \xrightarrow{\delta_{m-1}^+} \dots \xrightarrow{\delta_{k_0+1}^+} C(\Gamma'_{k_0}) \rightarrow 0,$$

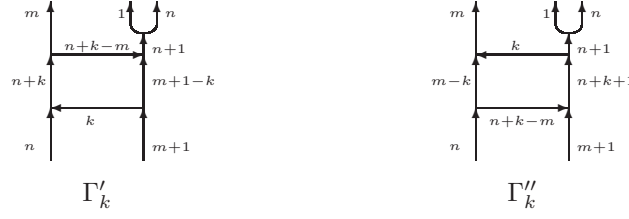


FIGURE 74.

and C^- the chain complex

$$(12.6) \quad 0 \rightarrow C(\Gamma'_{k_0}) \xrightarrow{\delta_{k_0}^-} \dots \xrightarrow{\delta_{m-2}^-} C(\Gamma'_{m-1}) \xrightarrow{P_{m-1,m-1}} C(\Gamma''_{m-1}) \rightarrow 0,$$

where $k_0 = \max\{m-n, 0\}$ and $J_{m-1,m-1}$, $P_{m-1,m-1}$ are defined in Definition 11.23. Then, by Lemma 11.19 and Proposition 11.25, both C^\pm are isomorphic in $\text{Ch}^b(\text{hmf})$ to

$$\bigoplus_{j=k_0}^{m-1} (0 \longrightarrow C(\Gamma''_j) \xrightarrow{\cong} C(\Gamma''_j) \longrightarrow 0,)$$

and are therefore homotopic to 0.



FIGURE 75.

Now consider $\hat{C}(D_{1,1}^\pm)$. Note that $D_{1,1}^\pm$ has two crossings – one $c_{m,n}^\pm$ and one $c_{1,n}^\pm$. Denote by d_k^\pm the differential map of the $c_{m,n}^\pm$ crossing. From Corollary 11.28, the differential map $c_{1,n}^+$ (resp. $c_{1,n}^-$) is χ^1 (resp. χ^0 .) Let $\Gamma_{k,0}$ and $\Gamma_{k,1}$ be the MOY graphs in Figure 75. Then d_k^\pm acts on the left square in $\Gamma_{k,0}$ and $\Gamma_{k,1}$, and χ^0 , χ^1 act on the upper right corners of $\Gamma_{k,0}$ and $\Gamma_{k,1}$. The chain complex $\hat{C}(D_{1,1}^+)$ is

$$(12.7) \quad 0 \rightarrow C(\Gamma_{m,1}) \xrightarrow{\mathfrak{d}_m^+} \begin{array}{c} C(\Gamma_{m,0})\{q^{-1}\} \\ \oplus \\ C(\Gamma_{m-1,1})\{q^{-1}\} \end{array} \xrightarrow{\mathfrak{d}_{m-1}^+ \dots \mathfrak{d}_{k+1}^+} \begin{array}{c} C(\Gamma_{k+1,0})\{q^{k-m}\} \\ \oplus \\ C(\Gamma_{k,1})\{q^{k-m}\} \end{array} \xrightarrow{\mathfrak{d}_k^+ \dots \mathfrak{d}_{k_0}^+} C(\Gamma_{k_0,0})\{q^{k_0-1-m}\} \rightarrow 0,$$

where $k_0 = \max\{m-n, 0\}$ as above and

$$\begin{aligned} \mathfrak{d}_m^+ &= \begin{pmatrix} \chi^1 \\ -d_m^+ \end{pmatrix}, \\ \mathfrak{d}_k^+ &= \begin{pmatrix} d_{k+1}^+ & \chi^1 \\ 0 & -d_k^+ \end{pmatrix} \quad \text{for } k_0 < k < m, \\ \mathfrak{d}_{k_0}^+ &= \begin{pmatrix} d_{k_0+1}^+ & \chi^1 \end{pmatrix}. \end{aligned}$$

Similarly, The chain complex $\hat{C}(D_{1,1}^-)$ is

(12.8)

$$0 \rightarrow C(\Gamma_{k_0,0})\{q^{m+1-k_0}\} \xrightarrow{\mathfrak{d}_{k_0}^-} \dots \xrightarrow{\mathfrak{d}_{k-1}^-} \begin{matrix} C(\Gamma_{k,0})\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma_{k-1,1})\{q^{m+1-k}\} \end{matrix} \xrightarrow{\mathfrak{d}_k^-} \dots \xrightarrow{\mathfrak{d}_{m-1}^-} \begin{matrix} C(\Gamma_{m,0})\{q\} \\ \oplus \\ C(\Gamma_{m-1,1})\{q\} \end{matrix} \xrightarrow{\mathfrak{d}_m^-} C(\Gamma_{m,1}) \rightarrow 0,$$

where $k_0 = \max\{m-n, 0\}$ as above and

$$\begin{aligned} \mathfrak{d}_{k_0}^- &= \begin{pmatrix} d_{k_0}^- \\ \chi_0^- \end{pmatrix}, \\ \mathfrak{d}_k^- &= \begin{pmatrix} d_k^- & 0 \\ \chi_0^- & -d_{k-1}^- \end{pmatrix} \quad \text{for } k_0 < k < m, \\ \mathfrak{d}_m^- &= \begin{pmatrix} \chi_0^- & -d_{m-1}^- \end{pmatrix}. \end{aligned}$$

Next, we study relations between the chain complexes $\hat{C}(D_{1,1}^\pm)$, $\hat{C}(D_{1,0}^\pm)$ and C^\pm .

12.2. Basic commutativity lemmas. To prove (12.2), we will frequently use the fact that certain morphisms of matrix factorizations of MOY graphs commute with each other. We establish two basic commutativity lemmas in this subsection.

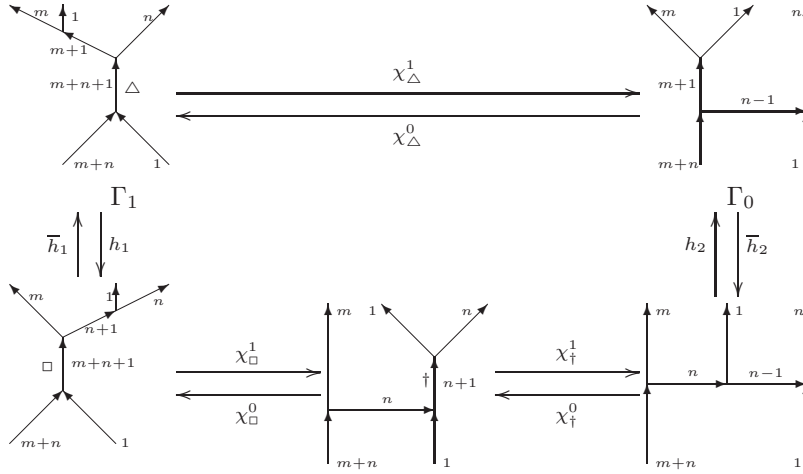


FIGURE 76.

Lemma 12.3. *Consider the diagram in Figure 76, where the morphisms are induced by the apparent local changes of MOY graphs. Then $\chi_\Delta^1 \approx h_2 \circ \chi_\dagger^1 \circ \chi_\square^1 \circ h_1$ and $\chi_\Delta^0 \approx \bar{h}_1 \circ \chi_\square^0 \circ \chi_\dagger^0 \circ \bar{h}_2$. That is, up to homotopy and scaling, the diagram in Figure 76 commutes in both directions.*

Proof. Denote by \bigcirc_{m+n+1} an oriented circle colored by $m+n+1$, and by Γ , Γ' and MOY graphs in Figure 77. Let $\bar{\Gamma}$ be Γ with the opposite orientation. Then, by

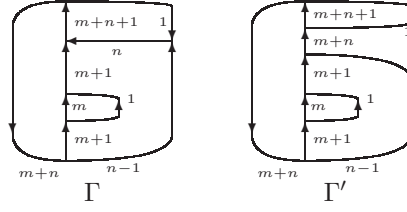


FIGURE 77.

Corollary 5.13, Theorem 5.14 and Corollary 6.1,

$$\begin{aligned}
 \text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0)) &\cong H(\Gamma) \langle m+n+1 \rangle \{q^{(m+n+1)(N-m-n-1)+2m+2n+mn}\} \\
 &\cong H(\Gamma') \langle m+n+1 \rangle \{q^{(m+n+1)(N-m-n-1)+2m+2n+mn}\} \\
 &\cong H(\bigcirc_{m+n+1}) \langle m+n+1 \rangle \{[m+1] \begin{bmatrix} m+n \\ m+1 \end{bmatrix} [m+n+1] q^{(m+n+1)(N-m-n-1)+2m+2n+mn}\} \\
 &\cong C(\emptyset) \{[m+1] \begin{bmatrix} m+n \\ m+1 \end{bmatrix} [m+n+1] \begin{bmatrix} N \\ m+n+1 \end{bmatrix} q^{(m+n+1)(N-m-n-1)+2m+2n+mn}\}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1)) &\cong H(\bar{\Gamma}) \langle m+n+1 \rangle \{q^{(m+n+1)(N-m-n-1)+2m+2n+mn}\} \\
 &\cong C(\emptyset) \{[m+1] \begin{bmatrix} m+n \\ m+1 \end{bmatrix} [m+n+1] \begin{bmatrix} N \\ m+n+1 \end{bmatrix} q^{(m+n+1)(N-m-n-1)+2m+2n+mn}\}.
 \end{aligned}$$

So $\text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0))$ and $\text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1))$ are concentrated on \mathbb{Z}_2 -degree 0, have lowest non-vanishing quantum grading $m+1$. And the subspaces of homogeneous elements of quantum degree $m+1$ of these spaces are 1-dimensional.

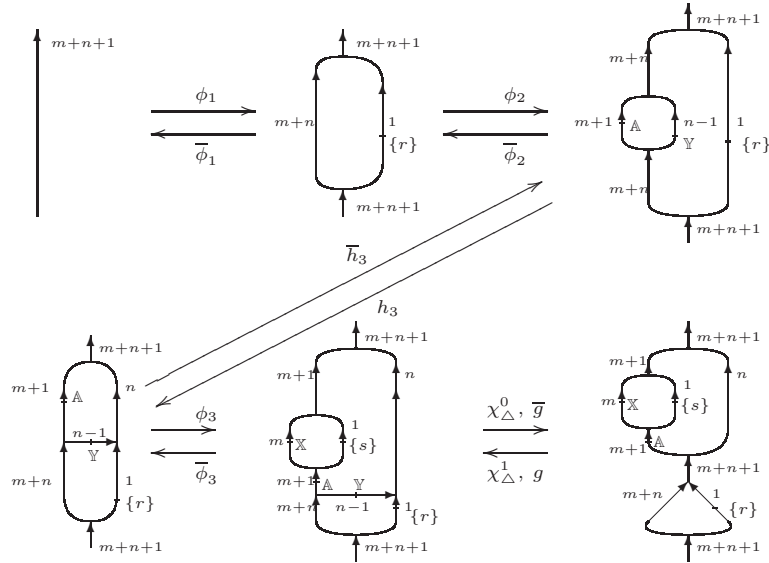


FIGURE 78.

Let $g = h_2 \circ \chi_{\dagger}^1 \circ \chi_{\square}^1 \circ h_1$ and $\bar{g} = \bar{h}_1 \circ \chi_{\square}^0 \circ \chi_{\dagger}^0 \circ \bar{h}_2$. To show that $\chi_{\Delta}^1 \approx g$ and $\chi_{\Delta}^0 \approx \bar{g}$, we only need to show that none of these morphisms are null-homotopic. For this purpose, consider the diagram in Figure 78, where ϕ_i , $\bar{\phi}_i$, h_3 and \bar{h}_3 are induced by the apparent local changes of MOY graphs.

Let $u = (-r)^{m+n}$, $v = S_{\lambda_{m+1, n-2}}(-\mathbb{Y})$ and $w = X_m$. Here X_j is the j -th elementary symmetric polynomial in \mathbb{X} . Then, by Corollary 9.3 and Lemma 7.11,

$$\begin{aligned}
& \bar{\phi}_1 \circ \mathbf{m}(u) \circ \bar{\phi}_2 \circ \mathbf{m}(v) \circ \bar{h}_3 \circ \bar{\phi}_3 \circ \mathbf{m}(w) \circ \chi_{\Delta}^1 \circ \chi_{\Delta}^0 \circ \phi_3 \circ h_3 \circ \phi_2 \circ \phi_1 \\
& \approx \bar{\phi}_1 \circ \mathbf{m}(u) \circ \bar{\phi}_2 \circ \mathbf{m}(v) \circ \bar{h}_3 \circ \bar{\phi}_3 \circ \mathbf{m}(w \sum_{k=0}^{m+1} (-r)^k A_{m+1-k}) \circ \phi_3 \circ h_3 \circ \phi_2 \circ \phi_1 \\
& \approx \bar{\phi}_1 \circ \mathbf{m}(u) \circ \bar{\phi}_2 \circ \mathbf{m}(v) \circ \mathbf{m}(\sum_{k=0}^{m+1} (-r)^k A_{m+1-k}) \circ \bar{h}_3 \circ \bar{\phi}_3 \circ \mathbf{m}(w) \circ \phi_3 \circ h_3 \circ \phi_2 \circ \phi_1 \\
& \approx \bar{\phi}_1 \circ \mathbf{m}(u) \circ \bar{\phi}_2 \circ \mathbf{m}(v \sum_{k=0}^{m+1} (-r)^k A_{m+1-k}) \circ \phi_2 \circ \phi_1 \\
& \approx \bar{\phi}_1 \circ \mathbf{m}(u) \circ (\mathbf{m}(\sum_{k=0}^{m+1} (-r)^k) \circ \bar{\phi}_2 \circ \mathbf{m}(v A_{m+1-k}) \circ \phi_2) \circ \phi_1 \\
& \approx \bar{\phi}_1 \circ \mathbf{m}(u) \circ \phi_1 \approx \text{id}_{C(\uparrow_{m+n+1})},
\end{aligned}$$

where A_j is the j -th elementary symmetric polynomial in \mathbb{A} . This shows that χ_{Δ}^1 and χ_{Δ}^0 are both homotopically non-trivial.

Note that, by Corollary 9.3,

$$\begin{aligned}
g \circ \bar{g} &= h_2 \circ \chi_{\dagger}^1 \circ \chi_{\square}^1 \circ h_1 \circ \bar{h}_1 \circ \chi_{\square}^0 \circ \chi_{\dagger}^0 \circ \bar{h}_2 \\
&\approx h_2 \circ \chi_{\dagger}^1 \circ \chi_{\square}^1 \circ \chi_{\square}^0 \circ \chi_{\dagger}^0 \circ \bar{h}_2 \\
&\approx h_2 \circ \mathbf{m}((s-r) \sum_{k=0}^m (-r)^k X_{m-k}) \circ \bar{h}_2 \\
&\approx h_2 \circ \mathbf{m}(\sum_{k=0}^{m+1} (-r)^k (X_{m+1-k} + s X_{m-k})) \circ \bar{h}_2 \\
&\approx \mathbf{m}(\sum_{k=0}^{m+1} (-r)^k A_{m+1-k}) \approx \chi_{\Delta}^1 \circ \chi_{\Delta}^0.
\end{aligned}$$

So, the above argument also implies that

$$\bar{\phi}_1 \circ \mathbf{m}(u) \circ \bar{\phi}_2 \circ \mathbf{m}(v) \circ \bar{h}_3 \circ \bar{\phi}_3 \circ \mathbf{m}(w) \circ g \circ \bar{g} \circ \phi_3 \circ h_3 \circ \phi_2 \circ \phi_1 \approx \text{id}_{C(\uparrow_{m+n+1})}.$$

This shows that g and \bar{g} are both homotopically non-trivial. \square

Before stating the second commutativity lemma, we introduce a shorthand notation, which will be used throughout the rest of this section.

Definition 12.4. Consider the morphisms in Figure 79, where ϕ and $\bar{\phi}$ are the morphisms induced by the apparent edge splitting and merging, h and \bar{h} are induced by the apparent bouquet moves. Define $\varphi := h \circ \phi$ and $\bar{\varphi} := \bar{\phi} \circ \bar{h}$.

By Corollary 5.13, Lemmas 7.9 and 7.11, it is easy to check that, up to homotopy and scaling, φ and $\bar{\varphi}$ are the unique homotopically non-trivial homogeneous

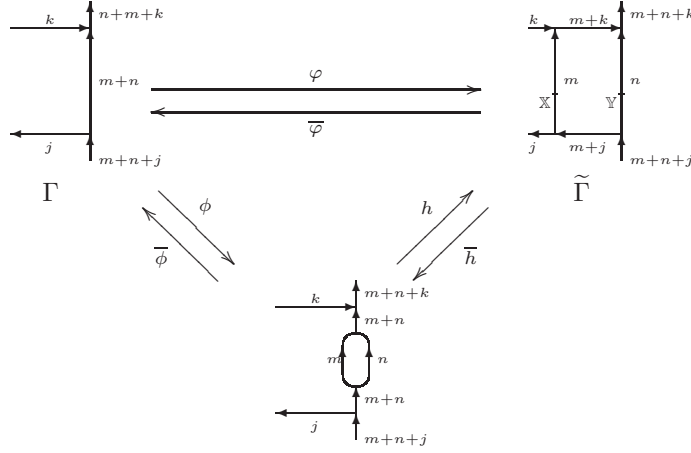


FIGURE 79.

morphisms between $C(\Gamma)$ and $C(\tilde{\Gamma})$ of \mathbb{Z}_2 -degree 0 and quantum degree $-mn$. And they satisfy, for $\lambda, \mu \in \Lambda_{m,n}$,

$$(12.9) \quad \overline{\varphi} \circ m(S_\lambda(\mathbb{X}) \cdot S_\mu(-\mathbb{Y})) \circ \varphi \approx \begin{cases} \text{id}_{C(\Gamma_0)} & \text{if } \lambda_i + \mu_{m+1-i} = n \ \forall i = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 12.5. *Consider the diagram in Figure 80, where φ_i and $\overline{\varphi}_i$ are the morphisms defined in Definition 12.4 associated to the apparent local changes of the MOY graphs. Then $\varphi_2 \circ \varphi_1 \approx \varphi_4 \circ \varphi_3$ and $\overline{\varphi}_1 \circ \overline{\varphi}_2 \approx \overline{\varphi}_3 \circ \overline{\varphi}_4$. That is, the diagram in Figure 80 commutes up to homotopy and scaling in both directions.*

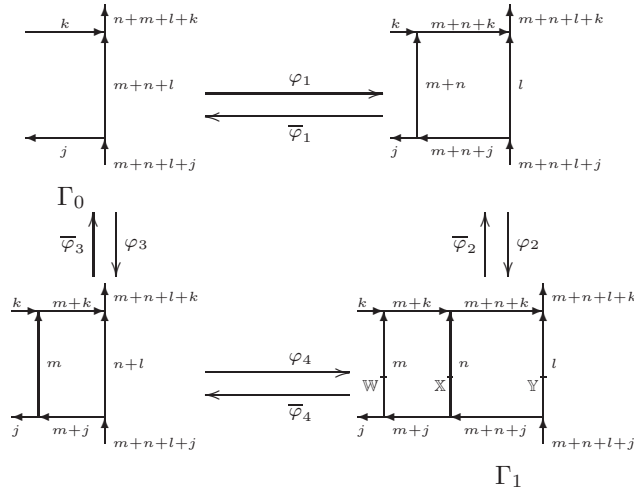


FIGURE 80.

Proof. By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have

$$C(\Gamma_1) \simeq C(\Gamma_0) \left\{ \begin{bmatrix} m+n+l \\ l \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix} \right\}.$$

So

$$\begin{aligned} \text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0)) &\cong \text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1)) \\ &\cong \text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_0)) \left\{ \begin{bmatrix} m+n+l \\ l \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix} \right\}. \end{aligned}$$

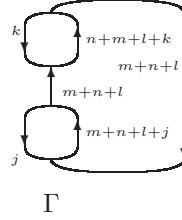


FIGURE 81.

Let Γ be the MOY graphs in Figure 81. Then

$$\begin{aligned} &\text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_0)) \\ &\cong H(\Gamma) \langle m+n+l+j+k \rangle \{ q^{(m+n+l+j+k)(N-m-n-l)-j^2-k^2} \} \\ &\cong C(\emptyset) \left\{ \begin{bmatrix} N-m-n-l \\ j \end{bmatrix} \begin{bmatrix} N-m-n-l \\ k \end{bmatrix} \begin{bmatrix} N \\ m+n+l \end{bmatrix} q^{(m+n+l+j+k)(N-m-n-l)-j^2-k^2} \right\}. \end{aligned}$$

So,

$$\begin{aligned} &\text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0)) \cong \text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1)) \\ &\cong C(\emptyset) \left\{ \begin{bmatrix} N-m-n-l \\ j \end{bmatrix} \begin{bmatrix} N-m-n-l \\ k \end{bmatrix} \begin{bmatrix} N \\ m+n+l \end{bmatrix} \begin{bmatrix} m+n+l \\ l \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix} q^{(m+n+l+j+k)(N-m-n-l)-j^2-k^2} \right\}. \end{aligned}$$

Thus, $\text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0))$ and $\text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1))$ are concentrated on \mathbb{Z}_2 -degree 0 and have lowest non-vanishing quantum grading $-mn - ml - nl$. And the subspaces of $\text{Hom}_{\text{HMF}}(C(\Gamma_1), C(\Gamma_0))$ and $\text{Hom}_{\text{HMF}}(C(\Gamma_0), C(\Gamma_1))$ of homogeneous elements of quantum grading $-mn - ml - nl$ are 1-dimensional.

$\varphi_2 \circ \varphi_1$, $\varphi_4 \circ \varphi_3$, $\overline{\varphi}_1 \circ \overline{\varphi}_2$ and $\overline{\varphi}_3 \circ \overline{\varphi}_4$ are all homogeneous of quantum degree $-mn - ml - nl$. So, to prove that $\varphi_2 \circ \varphi_1 \approx \varphi_4 \circ \varphi_3$ and $\overline{\varphi}_1 \circ \overline{\varphi}_2 \approx \overline{\varphi}_3 \circ \overline{\varphi}_4$, we only need to show that $\varphi_2 \circ \varphi_1$, $\varphi_4 \circ \varphi_3$, $\overline{\varphi}_1 \circ \overline{\varphi}_2$ and $\overline{\varphi}_3 \circ \overline{\varphi}_4$ are homotopically non-trivial. For this purpose, consider equation (12.9) above. We get

$$\begin{aligned} &\overline{\varphi}_1 \circ \overline{\varphi}_2 \circ \mathbf{m}(S_{\lambda_{n,m}}(\mathbb{X}) \cdot S_{\lambda_{l,m+n}}(\mathbb{Y})) \circ \varphi_2 \circ \varphi_1 \\ &\simeq \overline{\varphi}_1 \circ \overline{\varphi}_2 \circ \mathbf{m}(S_{\lambda_{n,m}}(\mathbb{X})) \circ \varphi_2 \circ \mathbf{m}(S_{\lambda_{l,m+n}}(\mathbb{Y})) \circ \varphi_1 \\ &\approx \text{id}_{C(\Gamma_0)}. \end{aligned}$$

This shows that $\varphi_2 \circ \varphi_1$ and $\overline{\varphi}_1 \circ \overline{\varphi}_2$ are homotopically non-trivial. Similarly,

$$\begin{aligned} &\overline{\varphi}_3 \circ \overline{\varphi}_4 \circ \mathbf{m}(S_{\lambda_{n,l}}(\mathbb{X}) \cdot S_{\lambda_{m,n+l}}(\mathbb{W})) \circ \varphi_4 \circ \varphi_3 \\ &\simeq \overline{\varphi}_3 \circ \overline{\varphi}_4 \circ \mathbf{m}(S_{\lambda_{n,l}}(\mathbb{X})) \circ \varphi_4 \circ \mathbf{m}(S_{\lambda_{m,n+l}}(\mathbb{W})) \circ \varphi_3 \\ &\approx \text{id}_{C(\Gamma_0)}. \end{aligned}$$

This shows that $\varphi_4 \circ \varphi_3$ and $\overline{\varphi}_3 \circ \overline{\varphi}_4$ are homotopically non-trivial. \square

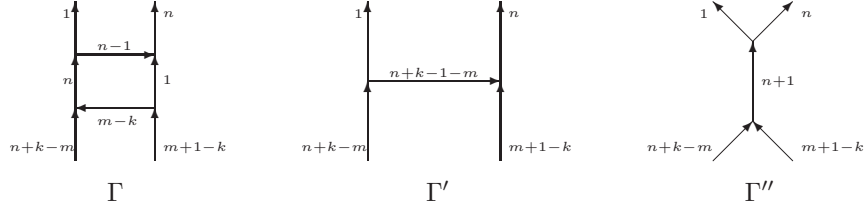


FIGURE 82.

12.3. Another look at Decomposition (IV). Decomposition (IV) (Theorem 9.1) plays an important role in relating $\hat{C}(D_{1,1}^\pm)$ to $\hat{C}(D_{1,0}^\pm)$ and C^\pm . In this subsection, we review a special case of Decomposition (IV), including the construction of all the morphisms involved.

Consider the MOY graphs in Figure 82. By Decomposition (IV) (Theorem 9.1), we have

$$(12.10) \quad C(\Gamma) \simeq C(\Gamma') \oplus C(\Gamma'')\{[m-k]\}.$$

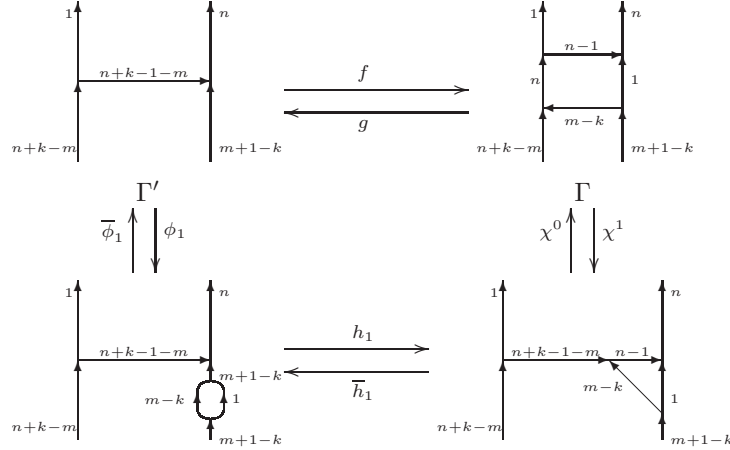


FIGURE 83.

By the construction in Subsection 9.1, especially Lemma 9.7, we know that the inclusion and projection between $C(\Gamma')$ and $C(\Gamma)$ are given by the compositions in Figure 83. That is, if

$$\begin{aligned} f &= \chi^0 \circ h_1 \circ \phi_1, \\ g &= \bar{\phi}_1 \circ \bar{h}_1 \circ \chi^1, \end{aligned}$$

where the morphisms on the right hand side are induced by the apparent local changes of MOY graphs, then f and g are homogeneous morphisms preserving both gradings and, after possibly a scaling,

$$g \circ f \simeq \text{id}_{C(\Gamma')}.$$

Lemma 12.6. *Diagrams (12.11) and (12.12) each give a pair of homogeneous homotopy equivalences preserving both gradings that are inverses of each other.*

Proof. We know that $C(\Gamma) \simeq C(\Gamma') \oplus C(\Gamma'')\{[m-k]\}$. So, to prove the lemma, we only need to show that

$$\begin{pmatrix} g \\ \tau \circ \vec{\beta} \end{pmatrix} \begin{pmatrix} f, & \vec{\alpha} \end{pmatrix} \simeq \begin{pmatrix} g \\ \vec{\beta} \end{pmatrix} \begin{pmatrix} f, & \vec{\alpha} \circ \tau \end{pmatrix} \simeq \text{id}_{C(\Gamma') \oplus C(\Gamma'')\{[m-k]\}}.$$

Consider $g \circ \vec{\alpha}$ and $\vec{\beta} \circ f$. By Lemma 9.14, we know that

$$\begin{cases} g \circ \mathbf{m}(r^j) \circ \alpha \simeq 0 & \text{if } j \leq m-k-1, \\ \beta \circ \mathbf{m}((-1)^{m-k-1-j} A_{m-k-1-j}) \circ f \simeq 0 & \text{if } j \geq 0. \end{cases}$$

This shows that

$$(12.13) \quad g \circ \vec{\alpha} \simeq 0 \text{ and } \vec{\beta} \circ f \simeq 0.$$

So,

$$\begin{aligned} \begin{pmatrix} g \\ \tau \circ \vec{\beta} \end{pmatrix} \begin{pmatrix} f, & \vec{\alpha} \end{pmatrix} &\simeq \begin{pmatrix} \text{id}_{C(\Gamma')} & g \circ \vec{\alpha} \\ \tau \circ \vec{\beta} \circ f & \text{id}_{C(\Gamma'')\{[m-k]\}} \end{pmatrix} \\ &\simeq \begin{pmatrix} \text{id}_{C(\Gamma')} & 0 \\ 0 & \text{id}_{C(\Gamma'')\{[m-k]\}} \end{pmatrix} \\ &= \text{id}_{C(\Gamma') \oplus C(\Gamma'')\{[m-k]\}}. \end{aligned}$$

And, similarly,

$$\begin{pmatrix} g \\ \vec{\beta} \end{pmatrix} \begin{pmatrix} f, & \vec{\alpha} \circ \tau \end{pmatrix} \simeq \text{id}_{C(\Gamma') \oplus C(\Gamma'')\{[m-k]\}}.$$

□

Next, we apply the above discussion to MOY graphs that appear in the chain complexes in Subsection 12.1.

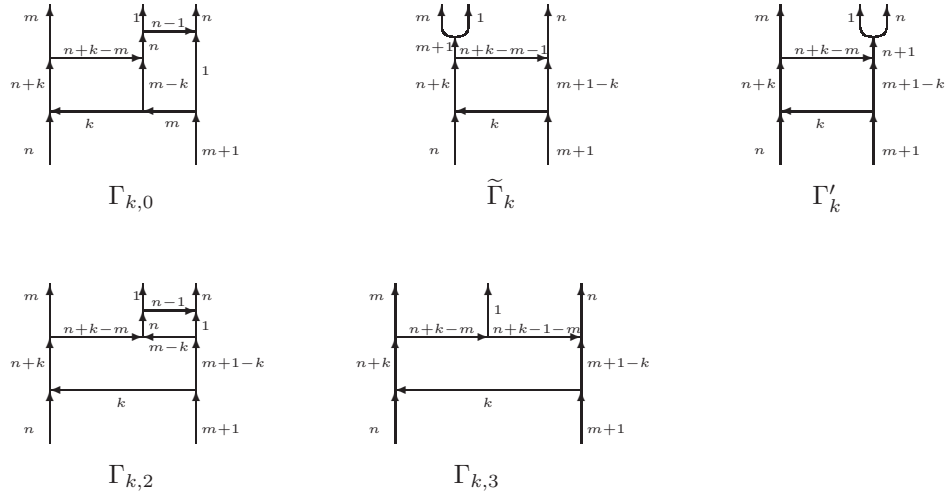


FIGURE 85.

Consider the MOY graphs in Figure 85. By Corollary 5.13, we have $C(\Gamma_{k,0}) \simeq C(\Gamma_{k,2})$ and $C(\tilde{\Gamma}_k) \simeq C(\Gamma_{k,3})$. By (12.10), $C(\Gamma_{k,2}) \simeq C(\Gamma_{k,3}) \oplus C(\Gamma'_k)\{[m-k]\}$. Altogether, we have

$$(12.14) \quad C(\Gamma_{k,0}) \simeq C(\tilde{\Gamma}_k) \oplus C(\Gamma'_k)\{[m-k]\}.$$

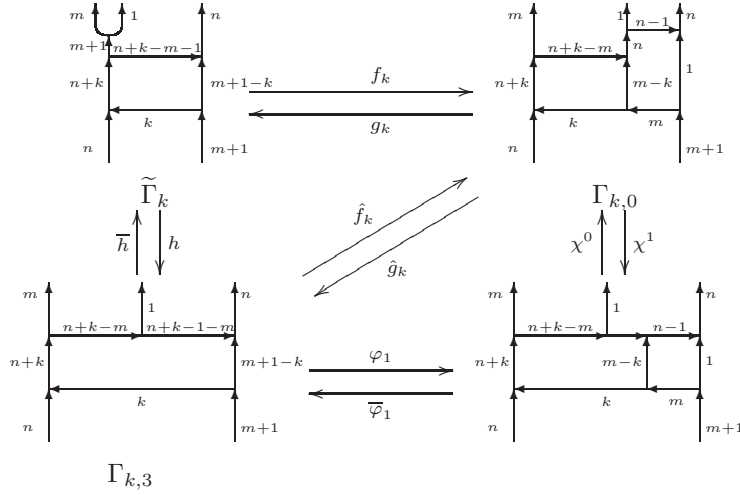


FIGURE 86.

In Figure 86, the morphism f_k , g_k , \hat{f}_k and \hat{g}_k are defined by

$$\begin{aligned} f_k &= \chi^0 \circ \varphi_1 \circ h, \\ g_k &= \bar{h} \circ \bar{\varphi}_1 \circ \chi^1, \\ \hat{f}_k &= \chi^0 \circ \varphi_1, \\ \hat{g}_k &= \bar{\varphi}_1 \circ \chi^1, \end{aligned}$$

where the morphisms on the right hand side are induced by the apparent local changes of MOY graphs. Then, after possibly a scaling,

$$(12.15) \quad g_k \circ f_k \simeq \text{id}_{C(\tilde{\Gamma}_k)}.$$

In Figure 87, the morphisms α_k and β_k are defined by

$$\begin{aligned} \alpha_k &= \chi^1 \circ \varphi_2, \\ \beta_k &= \bar{\varphi}_2 \circ \chi^0, \end{aligned}$$

where the morphisms on the right hand side are induced by the apparent local changes of MOY graphs. Define

$$\begin{aligned} \vec{\alpha}_k &= \sum_{j=0}^{m-k-1} \mathbf{m}(r^j) \circ \alpha_k = (\alpha_k, \mathbf{m}(r) \circ \alpha_k, \dots, \mathbf{m}(r^{m-k-1}) \circ \alpha_k), \\ \vec{\beta}_k &= \bigoplus_{j=0}^{m-k-1} \beta_k \circ \mathbf{m}((-1)^{m-k-1-j} A_{m-k-1-j}) = \begin{pmatrix} \beta_k \circ \mathbf{m}((-1)^{m-k-1} A_{m-k-1}) \\ \vdots \\ \beta_k \circ \mathbf{m}(-A_1) \\ \beta_k \end{pmatrix}. \end{aligned}$$

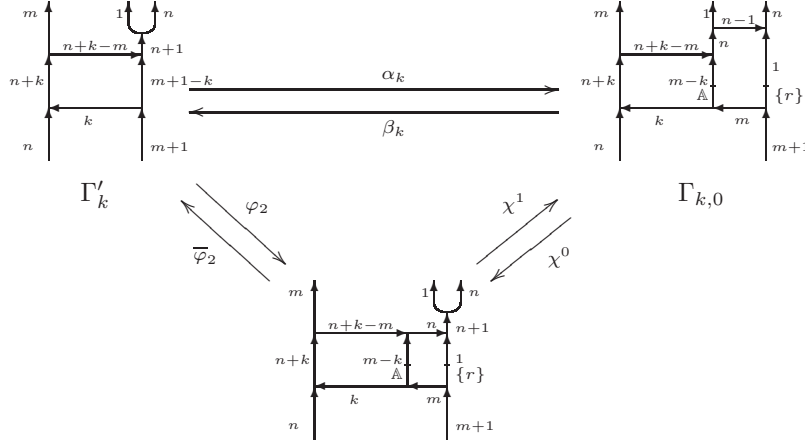


FIGURE 87.

Then there is a homogeneous morphism $\tau_k : C(\Gamma'_k)\{[m-k]\} \rightarrow C(\Gamma'_k)\{[m-k]\}$ preserving both gradings such that

$$(12.16) \quad \tau_k \circ \vec{\beta}_k \circ \vec{\alpha}_k \simeq \vec{\beta}_k \circ \vec{\alpha}_k \circ \tau_k \simeq \text{id}_{C(\Gamma'_k)\{[m-k]\}}.$$

We also have

$$(12.17) \quad g_k \circ \vec{\alpha}_k \simeq 0 \text{ and } \vec{\beta}_k \circ f_k \simeq 0.$$

From Lemma 12.6, we get the following corollary.

Corollary 12.7.

$$(12.18) \quad C(\Gamma_{k,0}) \begin{array}{c} \xrightarrow{\begin{pmatrix} g_k \\ \tau_k \circ \vec{\beta}_k \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} f_k, & \vec{\alpha}_k \end{pmatrix}} \end{array} \begin{array}{c} C(\tilde{\Gamma}_k) \\ \oplus \\ C(\Gamma'_k)\{[m-k]\} \end{array}$$

$$(12.19) \quad C(\Gamma_k) \begin{array}{c} \xrightarrow{\begin{pmatrix} g_k \\ \vec{\beta}_k \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} f_k, & \vec{\alpha}_k \circ \tau_k \end{pmatrix}} \end{array} \begin{array}{c} C(\tilde{\Gamma}_k) \\ \oplus \\ C(\Gamma'_k)\{[m-k]\} \end{array}$$

are two ways to explicitly write down the morphisms in the homotopy equivalence (12.14).

12.4. Relating the differential maps of C^\pm and $\hat{C}(D_{1,1}^\pm)$. Consider the diagram in Figure 88, where d_k^+ , d_{k-1}^- act on the left square, and φ_i , $\bar{\varphi}_i$ are induced by the apparent local changes of MOY graphs. We have the following lemma, which relates the differential maps of C^\pm and $\hat{C}(D_{1,1}^\pm)$.

Lemma 12.8. $\delta_k^+ \approx \bar{\varphi}_2 \circ d_k^+ \circ \mathbf{m}(r^{m-k}) \circ \varphi_1$ and $\delta_{k-1}^- \approx \bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ d_{k-1}^- \circ \varphi_2$. That is, the diagram in Figure 88 commutes up to homotopy and scaling in both directions.

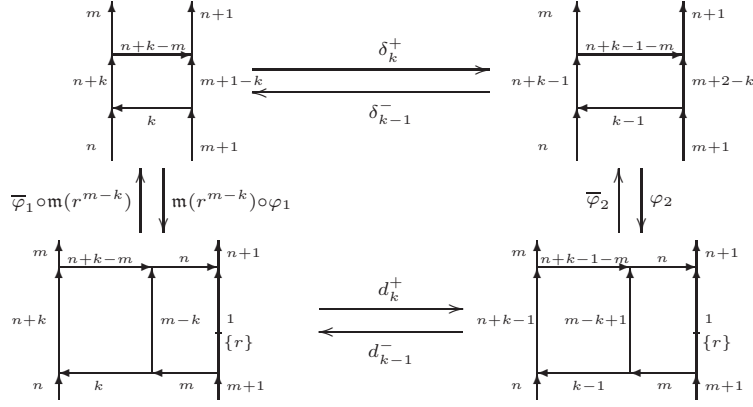


FIGURE 88.

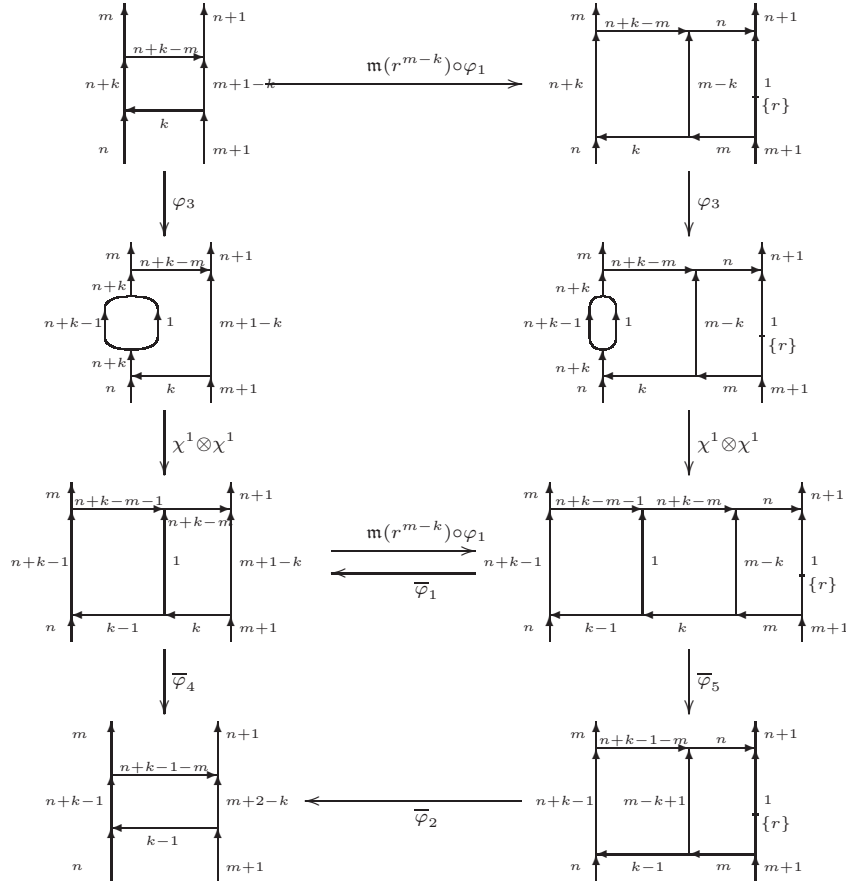


FIGURE 89.

Proof. Consider the diagram in Figure 89, where the morphisms are induced by the apparent local changes of MOY graphs. By Theorem 11.26, the composition of

the left column gives δ_k^+ and the composition of the right column gives d_k^+ . Since $(\chi^1 \otimes \chi^1) \circ \varphi_3$ and $\mathbf{m}(r^{m-k}) \circ \varphi_1$ act on different parts of the MOY graphs, they commute with each other. So

$$\mathbf{m}(r^{m-k}) \circ \varphi_1 \circ (\chi^1 \otimes \chi^1) \circ \varphi_3 \approx (\chi^1 \otimes \chi^1) \circ \varphi_3 \circ \mathbf{m}(r^{m-k}) \circ \varphi_1.$$

That is, the upper square in Figure 89 commutes up to homotopy and scaling. By Lemma 12.5, the lower square in Figure 89 commutes up to homotopy and scaling, i.e. $\overline{\varphi}_4 \circ \overline{\varphi}_1 \approx \overline{\varphi}_2 \circ \overline{\varphi}_5$. Recall that, by Lemma 7.11, we have $\overline{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ \varphi_1 \approx \text{id}$. Altogether, we have

$$\begin{aligned} \delta_k^+ &\approx \overline{\varphi}_4 \circ (\chi^1 \otimes \chi^1) \circ \varphi_3 \\ &\approx \overline{\varphi}_4 \circ \overline{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ \varphi_1 \circ (\chi^1 \otimes \chi^1) \circ \varphi_3 \\ &\approx \overline{\varphi}_2 \circ \overline{\varphi}_5 \circ (\chi^1 \otimes \chi^1) \circ \varphi_3 \circ \mathbf{m}(r^{m-k}) \circ \varphi_1 \\ &\approx \overline{\varphi}_2 \circ d_k^+ \circ \mathbf{m}(r^{m-k}) \circ \varphi_1. \end{aligned}$$

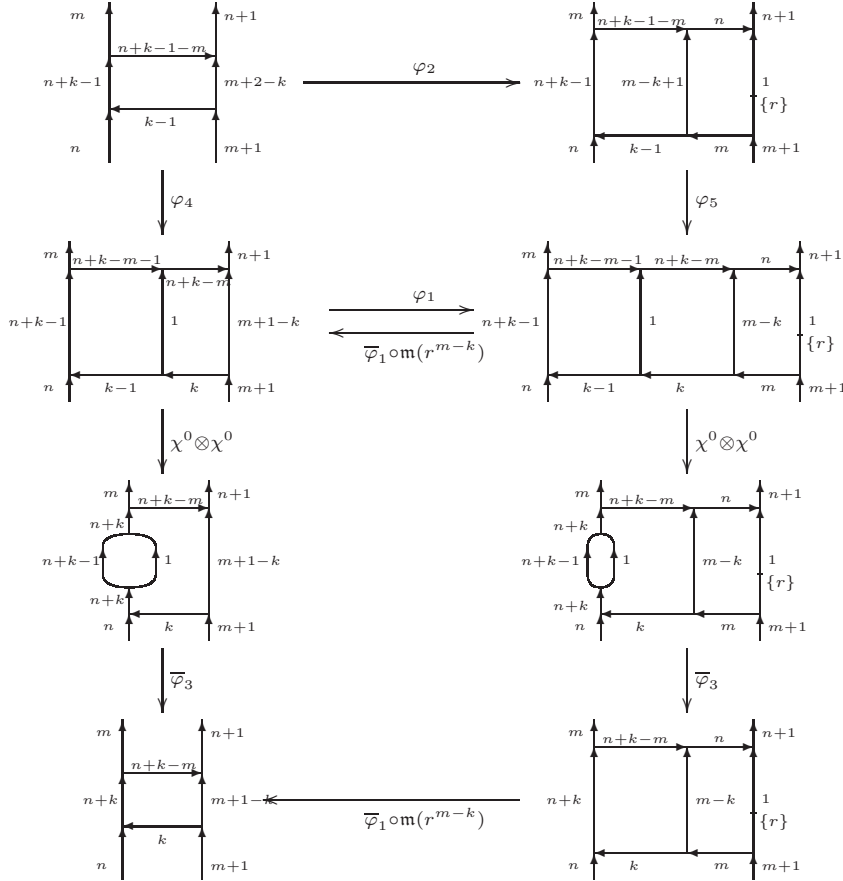


FIGURE 90.

Similarly, consider the diagram in Figure 90, where the morphisms are induced by the apparent local changes of MOY graphs. By Theorem 11.26, the composition of the left column gives δ_{k-1}^- and the composition of the right column gives d_{k-1}^- .

Since $\bar{\varphi}_3 \circ (\chi^0 \otimes \chi^0)$ and $\bar{\varphi}_1 \circ \mathbf{m}(r^{m-k})$ act on different parts of the MOY graphs, they commute with each other. So

$$\bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ \bar{\varphi}_3 \circ (\chi^0 \otimes \chi^0) \approx \bar{\varphi}_3 \circ (\chi^0 \otimes \chi^0) \circ \bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}).$$

That is, the lower square in Figure 90 commutes up to homotopy and scaling. By Lemma 12.5, the upper square in Figure 90 commutes up to homotopy and scaling, i.e. $\varphi_1 \circ \varphi_4 \approx \varphi_5 \circ \varphi_2$. Again, we have $\bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ \varphi_1 \approx \text{id}$. Altogether, we have

$$\begin{aligned} \delta_{k-1}^- &\approx \bar{\varphi}_3 \circ (\chi^0 \otimes \chi^0) \circ \varphi_4 \\ &\approx \bar{\varphi}_3 \circ (\chi^0 \otimes \chi^0) \circ \bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ \varphi_1 \circ \varphi_4 \\ &\approx \bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ \bar{\varphi}_3 \circ (\chi^0 \otimes \chi^0) \circ \varphi_5 \circ \varphi_2 \\ &\approx \bar{\varphi}_1 \circ \mathbf{m}(r^{m-k}) \circ d_{k-1}^- \circ \varphi_2. \end{aligned}$$

□

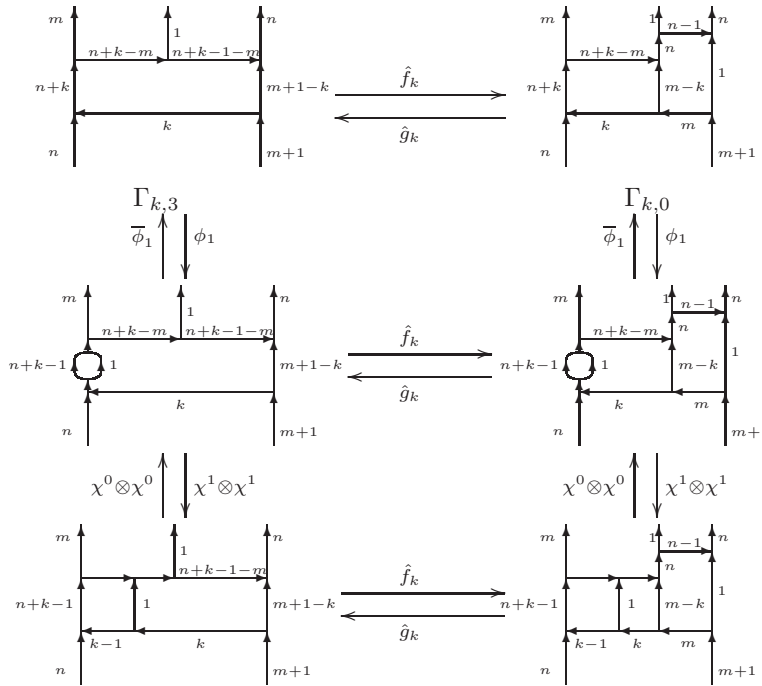


FIGURE 91.

12.5. Relating the differential maps of $\hat{C}(D_{1,0}^\pm)$ and $\hat{C}(D_{1,1}^\pm)$. Consider the diagram in Figure 91, where \hat{f}_k and \hat{g}_k are defined in Figure 86 and the vertical morphisms are induced by the apparent local changes of MOY graphs. Note that \hat{f}_k and \hat{g}_k act on the right side of the MOY graphs only, and the vertical morphisms act on the left side only. So each square in the Figure 91 commutes in both directions up to homotopy and scaling. Thus, we have the following lemma.

Lemma 12.9. *In Figure 91, we have $\hat{f}_k \circ (\chi^1 \otimes \chi^1) \circ \phi_1 \approx (\chi^1 \otimes \chi^1) \circ \phi_1 \circ \hat{f}_k$ and $\hat{g}_k \circ \bar{\phi}_1 \circ (\chi^0 \otimes \chi^0) \approx \bar{\phi}_1 \circ (\chi^0 \otimes \chi^0) \circ \hat{g}_k$.*

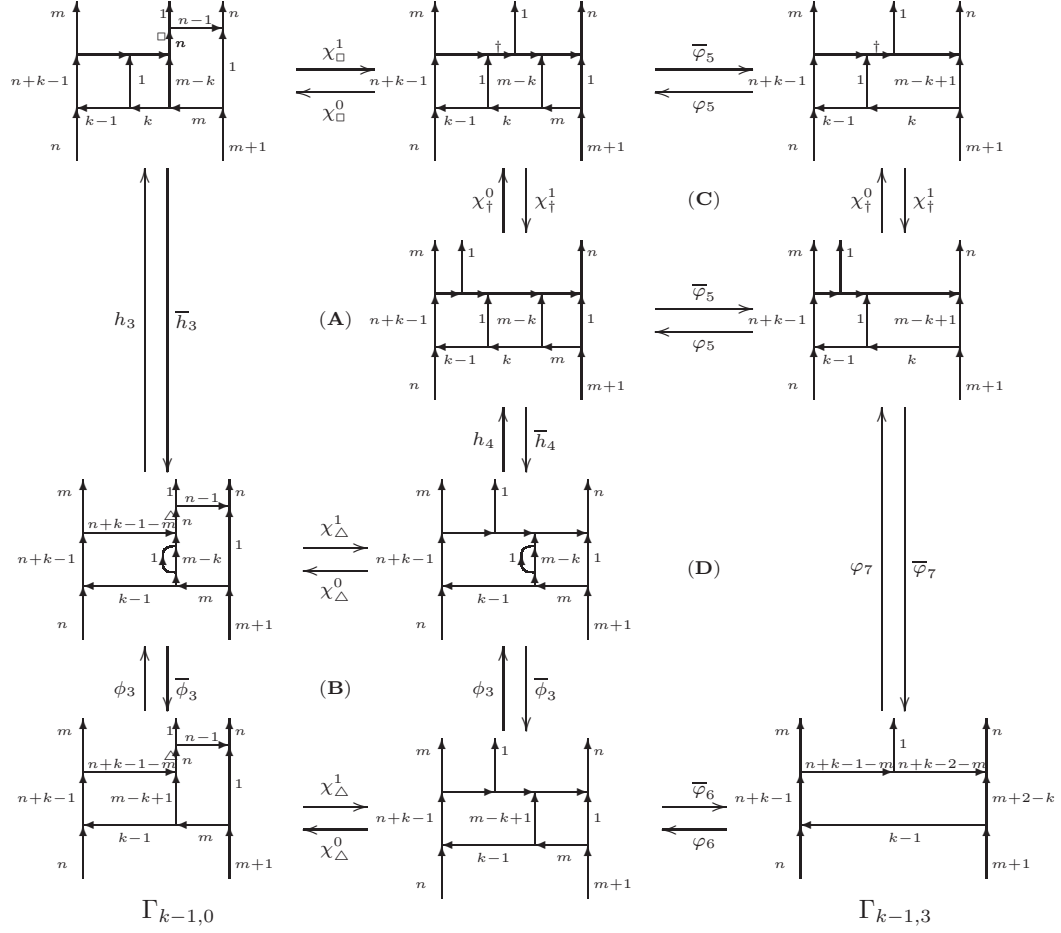


FIGURE 92.

Next, consider the diagram in Figure 92, where all morphisms are induced by the apparent local changes of the MOY graphs. We have the following lemma.

Lemma 12.10. *The four squares (A), (B), (C) and (D) in Figure 92 all commute up to homotopy and scaling in both directions. More specifically, we have*

- (A) $\chi^1_\Delta \circ \bar{h}_3 \approx \bar{h}_4 \circ \chi^1_\dagger \circ \chi^1_\square, h_3 \circ \chi^0_\Delta \approx \chi^0_\square \circ \chi^0_\dagger \circ h_4,$
- (B) $\chi^1_\Delta \circ \bar{\phi}_3 \approx \bar{\phi}_3 \circ \chi^1_\Delta, \phi_3 \circ \chi^0_\Delta \approx \chi^0_\Delta \circ \phi_3,$
- (C) $\bar{\varphi}_5 \circ \chi^1_\dagger \approx \bar{\varphi}_5 \circ \chi^1_\dagger, \chi^0_\dagger \circ \varphi_5 \approx \varphi_5 \circ \chi^0_\dagger,$
- (D) $\bar{\varphi}_7 \circ \bar{\varphi}_5 \approx \bar{\varphi}_6 \circ \bar{\phi}_3 \circ \bar{h}_4, \varphi_5 \circ \varphi_7 \approx h_4 \circ \phi_3 \circ \varphi_6.$

Altogether, we have

$$\begin{aligned} \bar{\varphi}_6 \circ \chi^1_\Delta \circ \bar{\phi}_3 \circ \bar{h}_3 &\approx \bar{\varphi}_7 \circ \chi^1_\dagger \circ \bar{\varphi}_5 \circ \chi^1_\square, \\ h_3 \circ \phi_3 \circ \chi^0_\Delta \circ \varphi_6 &\approx \chi^0_\square \circ \varphi_5 \circ \chi^0_\dagger \circ \varphi_7. \end{aligned}$$

Proof. Part (A) follows from Lemma 12.3. (B) and (C) are true because the horizontal and vertical morphisms act on different parts of the MOY graphs. Part (D) follows from Lemma 12.5. \square

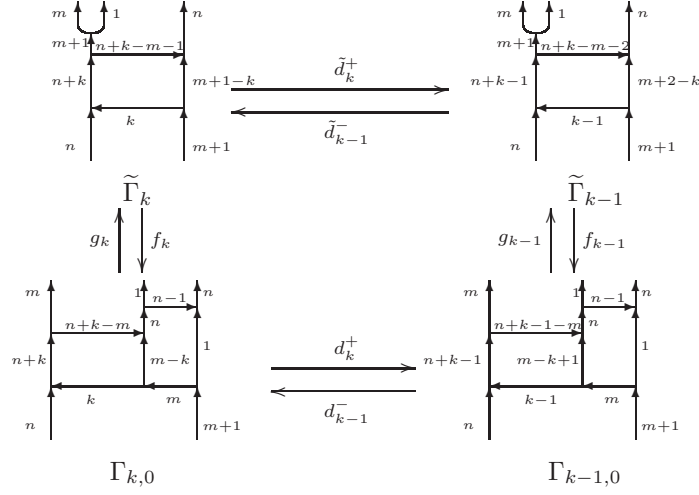


FIGURE 93.

We are now ready to relate the differential maps of $\hat{C}(D_{1,0}^\pm)$ and $\hat{C}(D_{1,1}^\pm)$. Consider the diagram in Figure 93, where d_\ast^\pm and \tilde{d}_\ast^\pm are defined in Subsection 12.1, and f_k, g_k are defined in Figure 86. We have the following lemma.

Lemma 12.11. *In Figure 93, $\tilde{d}_k^+ \approx g_{k-1} \circ d_k^+ \circ f_k$ and $\tilde{d}_{k-1}^- \approx g_k \circ d_{k-1}^- \circ f_{k-1}$. That is, the diagram in Figure 93 commutes in both directions up to homotopy and scaling.*

Proof. Denote by $h^{(k)}, \bar{h}^{(k)}, (\chi^1 \otimes \chi^1)^{(k)}$ and $(\chi^0 \otimes \chi^0)^{(k)}$ the morphisms induced by the local changes of MOY graphs in Figures 94 and 95. By the definitions of f_k, g_k, \hat{f}_k and \hat{g}_k in Figure 86, we know that $f_k \approx \hat{f}_k \circ h^{(k)}$ and $g_k \approx \bar{h}^{(k)} \circ \hat{g}_k$.

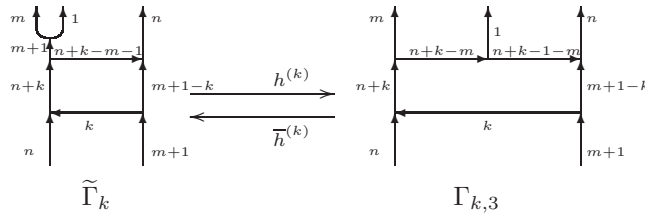


FIGURE 94.

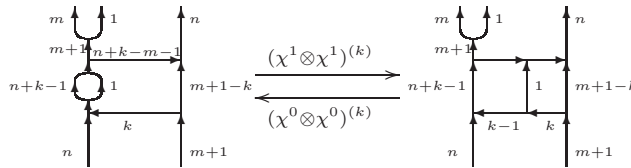


FIGURE 95.

By the definitions of \hat{f}_k , \hat{g}_k and d_k^+ , using the morphisms in Figures 91 and 92, we have

$$\begin{aligned}
g_{k-1} \circ d_k^+ \circ f_k &\approx \bar{h}^{(k-1)} \circ \hat{g}_{k-1} \circ d_k^+ \circ \hat{f}_k \circ h^{(k)} \\
&\approx \bar{h}^{(k-1)} \circ (\bar{\varphi}_6 \circ \chi_\Delta^1) \circ (\bar{\phi}_3 \circ \bar{h}_3 \circ (\chi^1 \otimes \chi^1) \circ \phi_1) \circ \hat{f}_k \circ h^{(k)} \\
&\approx \bar{h}^{(k-1)} \circ (\bar{\varphi}_6 \circ \chi_\Delta^1 \circ \bar{\phi}_3 \circ \bar{h}_3) \circ ((\chi^1 \otimes \chi^1) \circ \phi_1) \circ \hat{f}_k \circ h^{(k)} \\
(\text{by Lemma 12.10}) &\approx \bar{h}^{(k-1)} \circ (\bar{\varphi}_7 \circ \chi_\dagger^1 \circ \bar{\varphi}_5 \circ \chi_\square^1) \circ ((\chi^1 \otimes \chi^1) \circ \phi_1) \circ \hat{f}_k \circ h^{(k)} \\
(\text{by Lemma 12.9}) &\approx \bar{h}^{(k-1)} \circ (\bar{\varphi}_7 \circ \chi_\dagger^1 \circ \bar{\varphi}_5 \circ \chi_\square^1) \circ \hat{f}_k \circ ((\chi^1 \otimes \chi^1) \circ \phi_1) \circ h^{(k)}.
\end{aligned}$$

Note that $\bar{\varphi}_5 \circ \chi_\square^1 \approx \hat{g}_k$, $\hat{g}_k \circ \hat{f}_k \approx \text{id}$ and $\phi_1 \circ h^{(k)} \approx h^{(k)} \circ \phi_1$. So, from above, we have

$$\begin{aligned}
g_{k-1} \circ d_k^+ \circ f_k &\approx \bar{h}^{(k-1)} \circ (\bar{\varphi}_7 \circ \chi_\dagger^1) \circ (\bar{\varphi}_5 \circ \chi_\square^1) \circ \hat{f}_k \circ (\chi^1 \otimes \chi^1) \circ \phi_1 \circ h^{(k)} \\
&\approx \bar{h}^{(k-1)} \circ \bar{\varphi}_7 \circ \chi_\dagger^1 \circ (\chi^1 \otimes \chi^1) \circ h^{(k)} \circ \phi_1
\end{aligned}$$

By Lemma 12.3, we know

$$\chi_\dagger^1 \circ (\chi^1 \otimes \chi^1) \circ h^{(k)} \approx h^{(k-1)} \circ (\chi^1 \otimes \chi^1)^{(k)}.$$

Also, it is easy to see that $\bar{h}^{(k-1)} \circ \bar{\varphi}_7 \approx \bar{\varphi}_7 \circ \bar{h}^{(k-1)}$. So

$$\begin{aligned}
g_{k-1} \circ d_k^+ \circ f_k &\approx \bar{h}^{(k-1)} \circ \bar{\varphi}_7 \circ \chi_\dagger^1 \circ (\chi^1 \otimes \chi^1) \circ h^{(k)} \circ \phi_1 \\
&\approx \bar{\varphi}_7 \circ \bar{h}^{(k-1)} \circ h^{(k-1)} \circ (\chi^1 \otimes \chi^1)^{(k)} \circ \phi_1 \\
&\approx \bar{\varphi}_7 \circ (\chi^1 \otimes \chi^1)^{(k)} \circ \phi_1 \\
&\approx \tilde{d}_k^+.
\end{aligned}$$

Similarly,

$$\begin{aligned}
g_k \circ d_{k-1}^- \circ f_{k-1} &\approx \bar{h}^{(k)} \circ \hat{g}_k \circ d_{k-1}^- \circ \hat{f}_{k-1} \circ h^{(k-1)} \\
&\approx \bar{h}^{(k)} \circ \hat{g}_k \circ (\bar{\phi}_1 \circ (\chi^0 \otimes \chi^0) \circ h_3 \circ \phi_3) \circ (\chi_\Delta^0 \circ \varphi_6) \circ h^{(k-1)} \\
(\text{by Lemma 12.9}) &\approx \bar{h}^{(k)} \circ (\bar{\phi}_1 \circ (\chi^0 \otimes \chi^0)) \circ \hat{g}_k \circ (h_3 \circ \phi_3 \circ \chi_\Delta^0 \circ \varphi_6) \circ h^{(k-1)} \\
(\text{by Lemma 12.10}) &\approx \bar{h}^{(k)} \circ (\bar{\phi}_1 \circ (\chi^0 \otimes \chi^0)) \circ \hat{g}_k \circ (\chi_\square^0 \circ \varphi_5 \circ \chi_\dagger^0 \circ \varphi_7) \circ h^{(k-1)} \\
(\text{since } \hat{g}_k \circ (\chi_\square^0 \circ \varphi_5) \approx \hat{g}_k \circ \hat{f}_k \approx \text{id}) &\approx \bar{\phi}_1 \circ \bar{h}^{(k)} \circ (\chi^0 \otimes \chi^0) \circ \chi_\dagger^0 \circ \varphi_7 \circ h^{(k-1)} \\
&\approx \bar{\phi}_1 \circ \bar{h}^{(k)} \circ (\chi^0 \otimes \chi^0) \circ \chi_\dagger^0 \circ h^{(k-1)} \circ \varphi_7 \\
(\text{by Lemma 12.3}) &\approx \bar{\phi}_1 \circ \bar{h}^{(k)} \circ h^{(k)} \circ (\chi^0 \otimes \chi^0)^{(k)} \circ \varphi_7 \\
&\approx \bar{\phi}_1 \circ (\chi^0 \otimes \chi^0)^{(k)} \circ \varphi_7 \\
&\approx \tilde{d}_{k-1}^-,
\end{aligned}$$

where we also used $\bar{\phi}_1 \circ \bar{h}^{(k)} \approx \bar{h}^{(k)} \circ \bar{\phi}_1$ and $\varphi_7 \circ h^{(k-1)} \approx h^{(k-1)} \circ \varphi_7$. \square

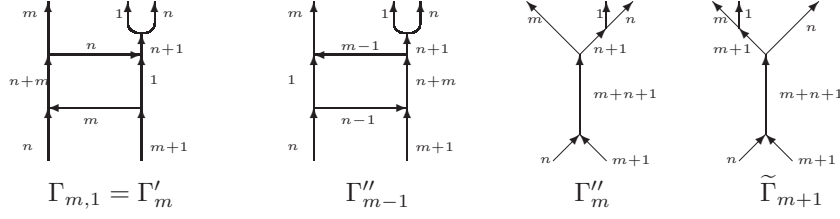


FIGURE 96.

12.6. **Decomposing** $C(\Gamma_{m,1}) = C(\Gamma'_m)$. Note that $\Gamma_{m,1}$ coincide with Γ'_m . Consider the MOY graphs in Figure 96. By Corollary 5.13, $C(\Gamma''_m) \simeq C(\tilde{\Gamma}_{m+1})$. By Decomposition (V) (Theorem 10.1), $C(\Gamma'_m) \simeq C(\Gamma''_{m-1}) \oplus C(\Gamma''_m)$. So

$$(12.20) \quad C(\Gamma_{m,1}) \simeq C(\Gamma''_{m-1}) \oplus C(\tilde{\Gamma}_{m+1}).$$

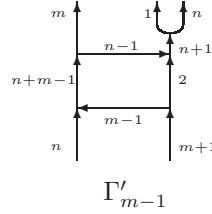


FIGURE 97.

Recall that Γ'_{m-1} is the MOY graph in Figure 97. We have the following lemma.

Lemma 12.12.

$$\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma'_{m-1})) \cong \text{Hom}_{\text{hmf}}(C(\Gamma'_{m-1}), C(\tilde{\Gamma}_{m+1})) \cong 0.$$

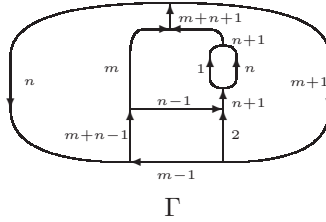


FIGURE 98.

Proof. Let Γ be the MOY graph in Figure 98. Recall that $C(\tilde{\Gamma}_{m+1}) \simeq C(\Gamma''_m)$. So

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma'_{m-1})) \cong \text{Hom}_{\text{HMF}}(C(\Gamma''_m), C(\Gamma'_{m-1})), \\ & \cong H(\Gamma) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2+n}\} \\ & \cong C(\emptyset) \{[n+1] \begin{bmatrix} n+m-1 \\ m \end{bmatrix} \begin{bmatrix} m+1 \\ 2 \end{bmatrix} \begin{bmatrix} m+n+1 \\ n \end{bmatrix} \begin{bmatrix} N \\ m+n+1 \end{bmatrix} q^{(m+n+1)(N-1)-m^2-n^2+n}\}. \end{aligned}$$

One can check that the lowest non-vanishing quantum gradings of the above space is 2. So $\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma''_{m-1})) \cong 0$.

Denote by $\bar{\Gamma}$ the MOY graph obtained by reversing the orientation of Γ . By Decomposition (V) (Theorem 10.1), we have $C(\Gamma'_{m-1}) \simeq C(\Gamma''_{m-1}) \oplus C(\Gamma''_{m-2})$. By Lemma 11.9, we have that

$$\text{Hom}_{\text{HMF}}(C(\Gamma''_k), C(\Gamma''_m)) \cong H(C(\Gamma''_m) \otimes C(\bar{\Gamma}''_k)) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2+n}\},$$

where $\bar{\Gamma}''_k$ is Γ''_k with reverse orientation, and the tensor is over the ring of partial symmetric polynomials in the alphabets marking the end points. Therefore,

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\Gamma'_{m-1}), C(\tilde{\Gamma}_{m+1})) \cong \text{Hom}_{\text{HMF}}(C(\Gamma'_{m-1}), C(\Gamma''_m)) \\ & \cong \text{Hom}_{\text{HMF}}(C(\Gamma''_{m-1}), C(\Gamma''_m)) \oplus \text{Hom}_{\text{HMF}}(C(\Gamma''_{m-2}), C(\Gamma''_m)) \\ & \cong (H(C(\Gamma''_m) \otimes C(\bar{\Gamma}''_{m-1})) \oplus H(C(\Gamma''_m) \otimes C(\bar{\Gamma}''_{m-2}))) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2+n}\} \\ & \cong H(C(\Gamma''_m) \otimes C(\bar{\Gamma}'_{m-1})) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2+n}\} \\ & \cong H(\bar{\Gamma}) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2+n}\} \\ & \cong C(\emptyset) \{[n+1] \begin{bmatrix} n+m-1 \\ m \end{bmatrix} \begin{bmatrix} m+1 \\ 2 \end{bmatrix} \begin{bmatrix} m+n+1 \\ n \end{bmatrix} \begin{bmatrix} N \\ m+n+1 \end{bmatrix} q^{(m+n+1)(N-1)-m^2-n^2+n}\}. \end{aligned}$$

where $\bar{\Gamma}'_{m-1}$ is Γ'_{m-1} with reverse orientation, and the tensor is over the ring of partial symmetric polynomials in the alphabets marking the end points. So the lowest non-vanishing quantum grading of $\text{Hom}_{\text{HMF}}(C(\Gamma'_{m-1}), C(\tilde{\Gamma}_{m+1}))$ is also 2. Thus, $\text{Hom}_{\text{hmf}}(C(\Gamma'_{m-1}), C(\tilde{\Gamma}_{m+1})) \cong 0$. \square

Corollary 12.13.

$$\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma''_{m-1})) \cong \text{Hom}_{\text{hmf}}(C(\Gamma''_{m-1}), C(\tilde{\Gamma}_{m+1})) \cong 0.$$

Proof. By Decomposition (V) (Theorem 10.1), $C(\Gamma'_{m-1}) \simeq C(\Gamma''_{m-1}) \oplus C(\Gamma''_{m-2})$. So $\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma''_{m-1}))$ (resp. $\text{Hom}_{\text{hmf}}(C(\Gamma''_{m-1}), C(\tilde{\Gamma}_{m+1}))$) is a subspace of $\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma'_{m-1}))$ (resp. $\text{Hom}_{\text{hmf}}(C(\Gamma'_{m-1}), C(\tilde{\Gamma}_{m+1}))$.) And the corollary follows from Lemma 12.12. \square

Lemma 12.14. $\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\tilde{\Gamma}_{m+1})) \cong \mathbb{C}$.

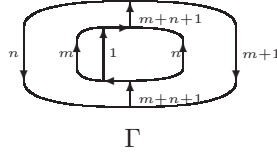


FIGURE 99.

Proof. Let Γ be the MOY graphs in Figure 99. Then

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\tilde{\Gamma}_{m+1}), C(\tilde{\Gamma}_{m+1})) \\ & \cong H(\Gamma) \langle m+n+1 \rangle \{q^{(m+n+1)(N-1)-m^2-n^2+n}\} \\ & \cong C(\emptyset) \{[m+1] \begin{bmatrix} m+n+1 \\ n \end{bmatrix} \begin{bmatrix} m+n+1 \\ n \end{bmatrix} \begin{bmatrix} N \\ m+n+1 \end{bmatrix} q^{(m+n+1)(N-1)-m^2-n^2+n}\} \end{aligned}$$

It is easy to check that the above space is concentrated on \mathbb{Z}_2 -degree 0. Its lowest non-vanishing quantum grading is 0. And its subspace of homogeneous elements of quantum degree 0 is 1-dimensional. Thus, $\text{Hom}_{\text{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\tilde{\Gamma}_{m+1})) \cong \mathbb{C}$. \square

Corollary 12.15.

$$\mathrm{Hom}_{\mathrm{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma_{m,1})) \cong \mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma_{m,1}), C(\tilde{\Gamma}_{m+1})) \cong \mathbb{C}.$$

Proof. This follows easily from (12.20), Corollary 12.13 and Lemma 12.14. \square

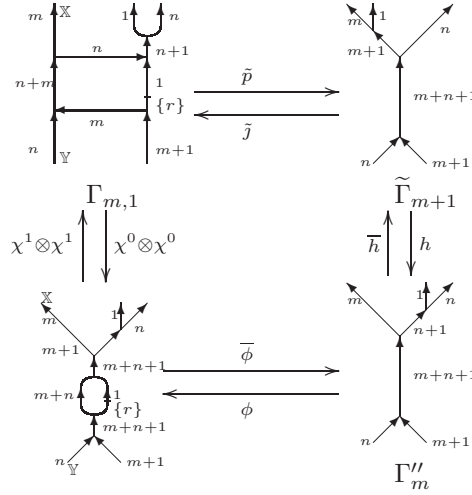


FIGURE 100.

Consider the diagram in Figure 100, where

$$\begin{aligned} \tilde{p} &:= \bar{h} \circ \bar{\phi} \circ (\chi^0 \otimes \chi^0), \\ \tilde{j} &:= (\chi^1 \otimes \chi^1) \circ \phi \circ h, \end{aligned}$$

and morphisms on the right hand side are induced by the apparent local changes of the MOY graphs and

Lemma 12.16. *Up to homotopy and scaling, \tilde{j} is the inclusion of $C(\tilde{\Gamma}_{m+1})$ into $C(\Gamma_{m,1})$ in (12.20), and \tilde{p} is the projection of $C(\Gamma_{m,1})$ onto $C(\tilde{\Gamma}_{m+1})$ in (12.20).*

Proof. From Corollary 12.15, one can see that $\mathrm{Hom}_{\mathrm{hmf}}(C(\tilde{\Gamma}_{m+1}), C(\Gamma_{m,1}))$ (resp. $\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma_{m,1}), C(\tilde{\Gamma}_{m+1}))$) is 1-dimensional and spanned by the inclusion $C(\tilde{\Gamma}_{m+1}) \rightarrow C(\Gamma_{m,1})$ (resp. the projection $C(\Gamma_{m,1}) \rightarrow C(\tilde{\Gamma}_{m+1})$) in (12.20). Note that \tilde{j} and \tilde{p} are both homogeneous morphisms of \mathbb{Z}_2 -degree 0 and quantum degree 0. To prove the lemma, we only need to show that \tilde{j} and \tilde{p} are not homotopic to 0. But, by Corollary 9.3 and Lemma 7.11,

$$\begin{aligned} \tilde{p} \circ \tilde{j} &\approx \bar{h} \circ \bar{\phi} \circ (\chi^0 \otimes \chi^0) \circ (\chi^1 \otimes \chi^1) \circ \phi \circ h \\ &\approx \bar{h} \circ \bar{\phi} \circ \mathbf{m}((\sum_{i=0}^n (-r)^i Y_{n-i}) \cdot (\sum_{i=0}^m (-r)^i X_{m-i})) \circ \phi \circ h \\ &\approx \bar{h} \circ \bar{\phi} \circ \mathbf{m}((-r)^{m+n}) \circ \phi \circ h \approx \mathrm{id}_{C(\tilde{\Gamma}_{m+1})}. \end{aligned}$$

This shows that \tilde{j} and \tilde{p} are not homotopic to 0 and proves the lemma. \square

Consider the diagram in Figure 101, where χ^0 , χ^1 , $h^{(m)}$ and $\bar{h}^{(m)}$ are induced by the apparent local changes of MOY graphs. We have the following lemma.

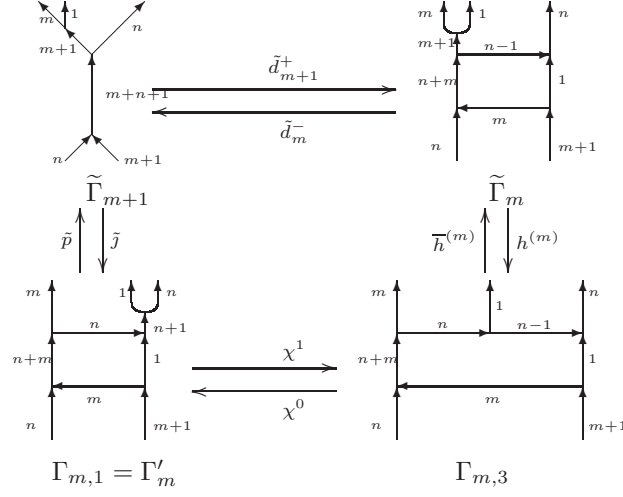


FIGURE 101.

Lemma 12.17. $\tilde{d}_{m+1}^+ \approx \bar{h}^{(m)} \circ \chi^1 \circ \tilde{j}$ and $\tilde{d}_m^- \approx \tilde{p} \circ \chi^0 \circ h^{(m)}$. That is, the diagram in Figure 101 commutes in both directions up to homotopy and scaling.

Proof. This follows easily from the definitions of \tilde{d}_{m+1}^+ , \tilde{d}_m^- , \tilde{j} , \tilde{p} and Lemma 12.3. \square

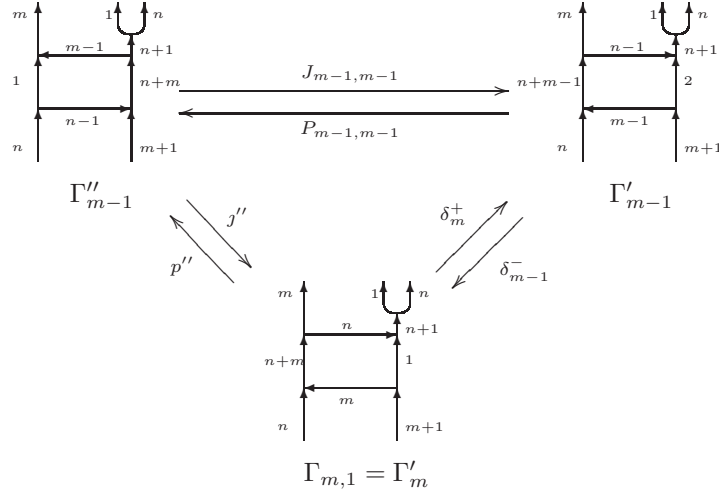


FIGURE 102.

Denote by $j'' : C(\Gamma_{m-1}'') \rightarrow C(\Gamma_{m,1})$ and $p'' : C(\Gamma_{m,1}) \rightarrow C(\Gamma_{m-1}'')$ the inclusion and projection between $C(\Gamma_{m-1}'')$ and $C(\Gamma_{m,1})$ in (12.20). Consider the diagram in Figure 102, where $J_{m-1,m-1}$, $P_{m-1,m-1}$ are defined in Definition 11.23. We have the following lemma.

Lemma 12.18. $\delta_m^+ \circ j'' \approx J_{m-1,m-1}$ and $p'' \circ \delta_{m-1}^- \approx P_{m-1,m-1}$. That is, the diagram in Figure 102 commutes in both directions up to homotopy and scaling.

Proof. Using Lemmas 11.9 and 11.19, one can check that

$$\mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma''_{m-1}), C(\Gamma'_{m-1})) \cong \mathrm{Hom}_{\mathrm{hmf}}(C(\Gamma'_{m-1}), C(\Gamma''_{m-1})) \cong \mathbb{C}.$$

Recall that $J_{m-1,m-1}$ and $P_{m-1,m-1}$ are both homogeneous morphisms of \mathbb{Z}_2 -degree 0 and quantum degree 0, and $P_{m-1,m-1} \circ J_{m-1,m-1} \approx \mathrm{id}_{C(\Gamma''_{m-1})}$. So $J_{m-1,m-1}$ and $P_{m-1,m-1}$ span these 1-dimensional spaces. Note that $\delta_m^+ \circ j''$ and $p'' \circ \delta_{m-1}^-$ are also homogeneous morphisms of \mathbb{Z}_2 -degree 0 and quantum degree 0. To prove the lemma, we only need to show that $\delta_m^+ \circ j''$ and $p'' \circ \delta_{m-1}^-$ are not homotopic to 0. But, by their definitions, we know that $p'' \circ \delta_{m-1}^- \circ \delta_m^+ \circ j'' \approx \mathrm{id}_{C(\Gamma''_{m-1})}$. So $\delta_m^+ \circ j''$ and $p'' \circ \delta_{m-1}^-$ are homotopically non-trivial. \square

12.7. Proof of Proposition 12.2. In this subsection, we prove (12.2), i.e.

$$\hat{C}(D_{10}^\pm) \simeq \hat{C}(D_{11}^\pm) \text{ if } l = 1.$$

The proof of the rest of Proposition 12.2 is very similar and left to the reader. We prove (12.2) by simplifying $\hat{C}(D_{11}^\pm)$ and reducing it to $\hat{C}(D_{10}^\pm)$. To do this, we need to use the following Gaussian Elimination Lemma, which is a version of [1, Lemma 4.2].

Lemma 12.19. [1, Lemma 4.2] *Let \mathcal{C} be an additive category, and*

$$\mathbf{I} = \cdots \rightarrow C \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} \begin{matrix} A \\ \oplus \\ D \end{matrix} \xrightarrow{\begin{pmatrix} \phi & \delta \\ \gamma & \varepsilon \end{pmatrix}} \begin{matrix} B \\ \oplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} \mu & \nu \end{pmatrix}} F \rightarrow \cdots$$

an object of $\mathrm{Ch}^b(\mathcal{C})$, that is, a bounded chain complex over \mathcal{C} . Assume that $A \xrightarrow{\phi} B$ is an isomorphism in \mathcal{C} with inverse ϕ^{-1} . Then \mathbf{I} is homotopic to (i.e. isomorphic in $\mathrm{hCh}^b(\mathcal{C})$ to)

$$\mathbf{II} = \cdots \rightarrow C \xrightarrow{\beta} D \xrightarrow{\varepsilon - \gamma\phi^{-1}\delta} E \xrightarrow{\nu} F \rightarrow \cdots.$$

In particular, if δ or γ is 0, then \mathbf{I} is homotopic to

$$\mathbf{II} = \cdots \rightarrow C \xrightarrow{\beta} D \xrightarrow{\varepsilon} E \xrightarrow{\nu} F \rightarrow \cdots.$$

Proof. Consider the chain complex

$$\mathbf{I}' = \cdots \rightarrow C \xrightarrow{\begin{pmatrix} 0 \\ \beta \end{pmatrix}} \begin{matrix} A \\ \oplus \\ D \end{matrix} \xrightarrow{\begin{pmatrix} \phi & 0 \\ 0 & \varepsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{matrix} B \\ \oplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} 0 & \nu \end{pmatrix}} F \rightarrow \cdots.$$

Define $f : \mathbf{I} \rightarrow \mathbf{I}'$ and $g : \mathbf{I}' \rightarrow \mathbf{I}$ by

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & C & \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} & \begin{matrix} A \\ \oplus \\ D \end{matrix} & \xrightarrow{\begin{pmatrix} \phi & \delta \\ \gamma & \varepsilon \end{pmatrix}} & \begin{matrix} B \\ \oplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} \mu & \nu \end{pmatrix}} F \longrightarrow \dots \\
 \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \begin{pmatrix} \text{id} & \phi^{-1}\delta \\ 0 & \text{id} \end{pmatrix} & & \downarrow \begin{pmatrix} \text{id} & 0 \\ -\gamma\phi^{-1} & \text{id} \end{pmatrix} \downarrow \text{id} & & \downarrow \text{id} \\
 \dots & \longrightarrow & C & \xrightarrow{\begin{pmatrix} 0 \\ \beta \end{pmatrix}} & \begin{matrix} A \\ \oplus \\ D \end{matrix} & \xrightarrow{\begin{pmatrix} \phi & 0 \\ 0 & \varepsilon - \gamma\phi^{-1}\delta \end{pmatrix}} & \begin{matrix} B \\ \oplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} 0 & \nu \end{pmatrix}} F \longrightarrow \dots \\
 \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \begin{pmatrix} \text{id} & -\phi^{-1}\delta \\ 0 & \text{id} \end{pmatrix} & & \downarrow \begin{pmatrix} \text{id} & 0 \\ \gamma\phi^{-1} & \text{id} \end{pmatrix} \downarrow \text{id} & & \downarrow \text{id} \\
 \dots & \longrightarrow & C & \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} & \begin{matrix} A \\ \oplus \\ D \end{matrix} & \xrightarrow{\begin{pmatrix} \phi & \delta \\ \gamma & \varepsilon \end{pmatrix}} & \begin{matrix} B \\ \oplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} \mu & \nu \end{pmatrix}} F \longrightarrow \dots
 \end{array}$$

It is easy to check that f and g are isomorphisms in $\text{Ch}^b(\mathcal{C})$. Thus,

$$\mathbf{I} \cong \mathbf{I}' \cong \mathbf{II} \oplus "0 \rightarrow A \xrightarrow{\phi} B \rightarrow 0".$$

But $0 \rightarrow A \xrightarrow{\phi} B \rightarrow 0$ is homotopic to 0 since ϕ is an isomorphism in \mathcal{C} . So $\mathbf{I} \simeq \mathbf{II}$. \square

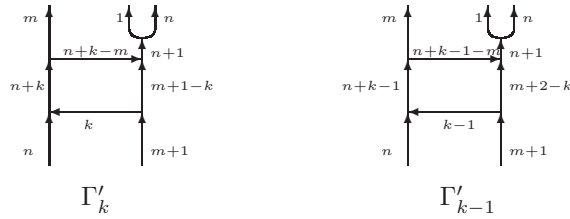


FIGURE 103.

Lemma 12.20.

$$\begin{aligned}
 \text{Hom}_{\text{hmf}}(C(\Gamma'_k)\{[m-k]q^{k-1-m}\}, C(\Gamma'_{k-1})) &\cong 0, \\
 \text{Hom}_{\text{hmf}}(C(\Gamma'_{k-1}), C(\Gamma'_k)\{[m-k]q^{m+1-k}\}) &\cong 0.
 \end{aligned}$$

Proof. By Decomposition (V) (more specifically, Lemma 11.19), we have that

$$C(\Gamma'_k) \simeq C(\Gamma''_k) \oplus C(\Gamma''_{k-1}).$$

Similar to Lemma 11.20, one can check that the lowest non-vanishing quantum grading of $\text{Hom}_{\text{HMF}}(C(\Gamma''_j), C(\Gamma''_k))$ is $(j-k)(j-k+1)$. So the lowest non-vanishing quantum grading of $\text{Hom}_{\text{HMF}}(C(\Gamma'_k), C(\Gamma'_{k-1}))$ and $\text{Hom}_{\text{HMF}}(C(\Gamma'_{k-1}), C(\Gamma'_k))$ is 0.

Note that

$$\begin{aligned}
& \text{Hom}_{\text{HMF}}(C(\Gamma'_k)\{[m-k]q^{k-1-m}\}, C(\Gamma'_{k-1})) \\
& \cong \text{Hom}_{\text{HMF}}(C(\Gamma'_k), C(\Gamma'_{k-1}))\{[m-k]q^{m+1-k}\} \\
& \cong \bigoplus_{j=0}^{m-1-k} \text{Hom}_{\text{HMF}}(C(\Gamma'_k), C(\Gamma'_{k-1})), \{q^{2+2j}\} \\
& \text{Hom}_{\text{HMF}}(C(\Gamma'_{k-1}), C(\Gamma'_k)\{[m-k]q^{m+1-k}\}) \\
& \cong \text{Hom}_{\text{HMF}}(C(\Gamma'_{k-1}), C(\Gamma'_k))\{[m-k]q^{m+1-k}\} \\
& \cong \bigoplus_{j=0}^{m-1-k} \text{Hom}_{\text{HMF}}(C(\Gamma'_{k-1}), C(\Gamma'_k))\{q^{2+2j}\},
\end{aligned}$$

and the lowest non-vanishing quantum grading of the right hand side is 2 in both cases. So

$$\begin{aligned}
\text{Hom}_{\text{hmf}}(C(\Gamma'_k)\{[m-k]q^{k-1-m}\}, C(\Gamma'_{k-1})) & \cong 0, \\
\text{Hom}_{\text{hmf}}(C(\Gamma'_{k-1}), C(\Gamma'_k)\{[m-k]q^{m+1-k}\}) & \cong 0.
\end{aligned}$$

□

We are now ready to prove (12.2). We prove $\hat{C}(D_{10}^+) \simeq \hat{C}(D_{11}^+)$ first and then $\hat{C}(D_{10}^-) \simeq \hat{C}(D_{11}^-)$.

Proof of $\hat{C}(D_{10}^+) \simeq \hat{C}(D_{11}^+)$ when $l = 1$. Recall that the chain complex $\hat{C}(D_{1,1}^+)$ is

$$0 \rightarrow C(\Gamma_{m,1}) \xrightarrow{\mathfrak{d}_m^+} \begin{array}{c} C(\Gamma_{m,0})\{q^{-1}\} \\ \oplus \\ C(\Gamma_{m-1,1})\{q^{-1}\} \end{array} \xrightarrow{\mathfrak{d}_{m-1}^+ \dots \mathfrak{d}_{k+1}^+} \begin{array}{c} C(\Gamma_{k+1,0})\{q^{k-m}\} \\ \oplus \\ C(\Gamma_{k,1})\{q^{k-m}\} \end{array} \xrightarrow{\mathfrak{d}_k^+ \dots \mathfrak{d}_{k_0}^+} C(\Gamma_{k_0,0})\{q^{k_0-1-m}\} \rightarrow 0,$$

where $k_0 = \max\{m-n, 0\}$ as above and

$$\begin{aligned}
\mathfrak{d}_m^+ &= \begin{pmatrix} \chi^1 \\ -d_m^+ \end{pmatrix}, \\
\mathfrak{d}_k^+ &= \begin{pmatrix} d_{k+1}^+ & \chi^1 \\ 0 & -d_k^+ \end{pmatrix} \quad \text{for } k_0 < k < m, \\
\mathfrak{d}_{k_0}^+ &= \begin{pmatrix} d_{k_0+1}^+ & \chi^1 \end{pmatrix}.
\end{aligned}$$

From Decomposition (IV) (more specifically, (12.14)), we have

$$C(\Gamma_{k,0}) \simeq C(\tilde{\Gamma}_k) \oplus C(\Gamma'_k)\{[m-k]\}.$$

By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have

$$C(\Gamma_{k,1}) \simeq C(\Gamma'_k)\{[m+1-k]\} \cong C(\Gamma'_k)\{q^{m-k}\} \oplus C(\Gamma'_k)\{[m-k]q^{-1}\}.$$

Therefore,

$$\begin{aligned}
& \begin{array}{c} C(\Gamma_{k+1,0})\{q^{k-m}\} \\ \oplus \\ C(\Gamma_{k,1})\{q^{k-m}\} \end{array} \simeq \begin{array}{c} C(\tilde{\Gamma}_{k+1})\{q^{k-m}\} \\ \oplus \\ C(\Gamma'_{k+1})\{[m-k-1]q^{k-m}\} \\ \oplus \\ C(\Gamma'_k) \\ \oplus \\ C(\Gamma'_k)\{[m-k]q^{k-m-1}\} \end{array} \quad \text{for } k_0 < k < m,
\end{aligned}$$

and

$$C(\Gamma_{k_0,0})\{q^{k_0-1-m}\} \simeq \frac{C(\tilde{\Gamma}_{k_0})\{q^{k_0-1-m}\}}{\oplus C(\Gamma'_{k_0})\{[m-k_0]q^{k_0-1-m}\}}.$$

So, $\hat{C}(D_{1,1}^+)$ is isomorphic to

$$0 \rightarrow C(\Gamma_{m,1}) \xrightarrow{\mathfrak{d}_m^+} \frac{C(\tilde{\Gamma}_m)\{q^{-1}\}}{\oplus C(\Gamma'_{m-1})\{q^{-2}\}} \xrightarrow{\mathfrak{d}_{m-1}^+ \dots \mathfrak{d}_{k+1}^+} \frac{C(\tilde{\Gamma}_{k+1})\{q^{k-m}\}}{\oplus C(\Gamma'_{k+1})\{[m-k-1]q^{k-m}\}} \xrightarrow{\mathfrak{d}_k^+ \dots \mathfrak{d}_{k_0}^+} \frac{C(\tilde{\Gamma}_{k_0})\{q^{k_0-1-m}\}}{\oplus C(\Gamma'_{k_0})\{[m-k_0]q^{k_0-1-m}\}} \rightarrow 0.$$

In this form, for $k_0 < k < m-1$, \mathfrak{d}_k^+ is given by a 4×4 matrix $(\mathfrak{d}_{k;i,j}^+)_{4 \times 4}$. Clearly,

$$\mathfrak{d}_{k;i,j}^+ = 0 \text{ for } (i,j) = (3,1), (3,2), (4,1), (4,2).$$

By Lemma 12.11,

$$\mathfrak{d}_{k;1,1}^+ \approx \tilde{d}_{k+1}^+.$$

By Lemma 12.8,

$$\mathfrak{d}_{k;3,3}^+ \approx \delta_k^+.$$

By (12.18) in Corollary 12.7 and that $g_k \circ \vec{\alpha}_k \simeq 0$, we know that

$$\begin{aligned} \mathfrak{d}_{k;1,4}^+ &\simeq 0, \\ \mathfrak{d}_{k;2,4}^+ &\approx \text{id}_{C(\Gamma'_k)\{[m-k]q^{k-m-1}\}}. \end{aligned}$$

By Lemma 12.20, we have

$$\mathfrak{d}_{k;3,4}^+ \simeq 0.$$

Altogether, we have that, for $k_0 < k < m-1$,

$$\mathfrak{d}_k^+ \simeq \begin{pmatrix} c_k \tilde{d}_{k+1}^+ & * & * & 0 \\ * & * & * & c_k'' \text{id}_{C(\Gamma'_k)\{[m-k]q^{k-m-1}\}} \\ 0 & 0 & c_k' \delta_k^+ & 0 \\ 0 & 0 & * & * \end{pmatrix},$$

where c_k , c_k' and c_k'' are non-zero scalars and $*$ means morphisms we have not determined. Similarly,

$$\begin{aligned} \mathfrak{d}_{k_0}^+ &\simeq \begin{pmatrix} c_{k_0} \tilde{d}_{k_0+1}^+ & * & * & 0 \\ * & * & * & c_{k_0}'' \text{id}_{C(\Gamma'_{k_0})\{[m-k_0+0]q^{k_0-1-m}\}} \end{pmatrix}, \\ \mathfrak{d}_{m-1}^+ &\simeq \begin{pmatrix} c_{m-1} \tilde{d}_m^+ & * & 0 \\ * & * & c_{m-1}'' \text{id}_{C(\Gamma'_{m-1})\{q^{-2}\}} \\ 0 & c_{m-1}' \delta_{m-1}^+ & 0 \\ 0 & * & * \end{pmatrix}, \end{aligned}$$

where c_{k_0} , c_{k_0}'' , c_{m-1} , c_{m-1}' and c_{m-1}'' are non-zero scalars.

Now apply Gaussian Elimination (Lemma 12.19) to $c_k'' \text{id}_{C(\Gamma'_k)\{[m-k]q^{k-m-1}\}}$ in \mathfrak{d}_k^+ for $k = k_0, k_0+1, \dots, m-1$ in that order. We get that $\hat{C}(D_{11}^+)$ is homotopic to

$$0 \rightarrow C(\Gamma_{m,1}) \xrightarrow{\hat{\mathfrak{d}}_m^+} \frac{C(\tilde{\Gamma}_m)\{q^{-1}\}}{\oplus C(\Gamma'_{m-1})} \xrightarrow{\hat{\mathfrak{d}}_{m-1}^+ \dots \hat{\mathfrak{d}}_{k+1}^+} \frac{C(\tilde{\Gamma}_{k+1})\{q^{k-m}\}}{\oplus C(\Gamma'_k)} \xrightarrow{\hat{\mathfrak{d}}_k^+ \dots \hat{\mathfrak{d}}_{k_0}^+} C(\tilde{\Gamma}_{k_0})\{q^{k_0-1-m}\} \rightarrow 0,$$

where

$$(12.21) \quad \hat{\mathfrak{d}}_k^+ \simeq \begin{pmatrix} c_k \tilde{d}_{k+1}^+ & * \\ 0 & c'_k \delta_k^+ \end{pmatrix} \text{ for } k_0 < k < m,$$

$$(12.22) \quad \hat{\mathfrak{d}}_{k_0}^+ \simeq \begin{pmatrix} c_{k_0} \tilde{d}_{k_0+1}^+ & * \\ c_{k_0} \tilde{d}_{k_0+1}^+ & * \end{pmatrix}.$$

Next we determine $\hat{\mathfrak{d}}_m^+$. By Decomposition (V) (more specifically, (12.20)), we have

$$C(\Gamma_{m,1}) \simeq \frac{C(\tilde{\Gamma}_{m+1})}{\oplus C(\Gamma''_{m-1})}.$$

Under this decomposition, $\hat{\mathfrak{d}}_m^+$ is represented by a 2×2 matrix. By Lemmas 12.12, 12.17 and 12.18, we know that

$$(12.23) \quad \hat{\mathfrak{d}}_m^+ \simeq \begin{pmatrix} c_m \tilde{d}_{m+1}^+ & * \\ 0 & c'_m J_{m-1,m-1} \end{pmatrix},$$

where c_m and c'_m are non-zero scalars. So $\hat{C}(D_{11}^+)$ is homotopic to

$$0 \rightarrow \frac{C(\tilde{\Gamma}_{m+1})}{\oplus C(\Gamma''_{m-1})} \xrightarrow{\hat{\mathfrak{d}}_m^+} \frac{C(\tilde{\Gamma}_m)\{q^{-1}\}}{\oplus C(\Gamma'_{m-1})} \xrightarrow{\hat{\mathfrak{d}}_{m-1}^+} \dots \xrightarrow{\hat{\mathfrak{d}}_{k+1}^+} \frac{C(\tilde{\Gamma}_{k+1})\{q^{k-m}\}}{\oplus C(\Gamma'_k)} \xrightarrow{\hat{\mathfrak{d}}_k^+} \dots \xrightarrow{\hat{\mathfrak{d}}_{k_0}^+} C(\tilde{\Gamma}_{k_0})\{q^{k_0-1-m}\} \rightarrow 0,$$

where $\hat{\mathfrak{d}}_m^+, \dots, \hat{\mathfrak{d}}_{k_0}^+$ are given in (12.21), (12.22) and (12.23).

Recall that, by Decomposition (V) (more specifically, Lemma 11.19),

$$C(\Gamma'_k) \simeq \begin{cases} C(\Gamma''_k) \oplus C(\Gamma''_{k-1}) & \text{if } k_0 + 1 \leq k \leq m-1, \\ C(\Gamma''_k) & \text{if } k = k_0. \end{cases}$$

By Proposition 11.25, under the decomposition

$$\frac{C(\tilde{\Gamma}_{k+1})\{q^{k-m}\}}{\oplus C(\Gamma'_k)} \simeq \frac{C(\tilde{\Gamma}_{k+1})\{q^{k-m}\}}{\oplus C(\Gamma''_k) \oplus C(\Gamma''_{k-1})},$$

we have

$$(12.24) \quad \hat{\mathfrak{d}}_m^+ \simeq \begin{pmatrix} c_m \tilde{d}_{m+1}^+ & * \\ 0 & c'''_m \text{id}_{C(\Gamma''_{m-1})} \\ 0 & 0 \end{pmatrix},$$

$$(12.25) \quad \hat{\mathfrak{d}}_k^+ \simeq \begin{pmatrix} c_k \tilde{d}_{k+1}^+ & * & * \\ 0 & 0 & c'''_k \text{id}_{C(\Gamma''_{k-1})} \\ 0 & 0 & 0 \end{pmatrix} \text{ for } k_0 + 1 < k < m,$$

where c'''_k is a non-zero scalar for $k_0 + 1 < k \leq m$. Since $C(\Gamma'_{k_0}) \simeq C(\Gamma''_{k_0})$, we have

$$\frac{C(\tilde{\Gamma}_{k_0+1})\{q^{k_0-m}\}}{\oplus C(\Gamma'_{k_0})} \simeq \frac{C(\tilde{\Gamma}_{k_0+1})\{q^{k_0-m}\}}{\oplus C(\Gamma''_{k_0})}$$

and

$$(12.26) \quad \hat{\mathfrak{d}}_{k_0+1}^+ \simeq \begin{pmatrix} c_{k_0+1}\tilde{d}_{k_0+2}^+ & * \\ 0 & 0 \end{pmatrix} c_{k_0+1}''' \text{id}_{C(\Gamma_{k_0}'')},$$

$$(12.27) \quad \hat{\mathfrak{d}}_{k_0}^+ \simeq \begin{pmatrix} c_{k_0}\tilde{d}_{k_0+1}^+ & * \end{pmatrix},$$

where c_{k_0+1}''' is a non-zero scalar. Putting these together, we know that $\hat{C}(D_{11}^+)$ is homotopic to

$$0 \rightarrow \begin{matrix} C(\tilde{\Gamma}_{m+1}) \\ \oplus \\ C(\Gamma_{m-1}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_m^+} \begin{matrix} C(\tilde{\Gamma}_m)\{q^{-1}\} \\ \oplus \\ C(\Gamma_{m-1}'') \\ \oplus \\ C(\Gamma_{m-2}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_{m-1}^+} \dots \xrightarrow{\hat{\mathfrak{d}}_{k+1}^+} \begin{matrix} C(\tilde{\Gamma}_{k+1})\{q^{k-m}\} \\ \oplus \\ C(\Gamma_k'') \\ \oplus \\ C(\Gamma_{k-1}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_k^+} \dots \xrightarrow{\hat{\mathfrak{d}}_{k_0+1}^+} \begin{matrix} C(\tilde{\Gamma}_{k_0+1})\{q^{k_0-m}\} \\ \oplus \\ C(\Gamma_{k_0}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_{k_0}^+} C(\tilde{\Gamma}_{k_0})\{q^{k_0-1-m}\} \rightarrow 0,$$

where $\hat{\mathfrak{d}}_m^+, \dots, \hat{\mathfrak{d}}_{k_0}^+$ are given in (12.24), (12.25), (12.26) and (12.27).

Applying Gaussian Elimination (Lemma 12.19) to $c_k''' \text{id}_{C(\Gamma_{k-1}'')}$ in $\hat{\mathfrak{d}}_k^+$ for $k = m, m-1, \dots, k_0+1$, we get that $\hat{C}(D_{11}^+)$ is homotopic to

$$0 \rightarrow C(\tilde{\Gamma}_{m+1}) \xrightarrow{\hat{\mathfrak{d}}_m^+} C(\tilde{\Gamma}_m)\{q^{-1}\} \xrightarrow{\hat{\mathfrak{d}}_{m-1}^+} \dots \xrightarrow{\hat{\mathfrak{d}}_{k+1}^+} C(\tilde{\Gamma}_{k+1})\{q^{k-m}\} \xrightarrow{\hat{\mathfrak{d}}_k^+} \dots \xrightarrow{\hat{\mathfrak{d}}_{k_0}^+} C(\tilde{\Gamma}_{k_0})\{q^{k_0-1-m}\} \rightarrow 0,$$

where $\tilde{\mathfrak{d}}_k^+ \simeq c_k \tilde{d}_{k+1}^+$ for $k = m, m-1, \dots, k_0$. Recall that $c_k \neq 0$ for $k = m, \dots, k_0$. So this last chain complex is isomorphic to $\hat{C}(D_{10}^+)$ in $\text{Ch}^b(\text{hmf})$. Therefore, $\hat{C}(D_{11}^+) \simeq \hat{C}(D_{10}^+)$. \square

Proof of $\hat{C}(D_{10}^-) \simeq \hat{C}(D_{11}^-)$ when $l = 1$. Recall that the chain complex $\hat{C}(D_{11}^-)$ is

$$0 \rightarrow C(\Gamma_{k_0,0})\{q^{m+1-k_0}\} \xrightarrow{\mathfrak{d}_{k_0}^-} \dots \xrightarrow{\mathfrak{d}_{k-1}^-} \begin{matrix} C(\Gamma_{k,0})\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma_{k-1,1})\{q^{m+1-k}\} \end{matrix} \xrightarrow{\mathfrak{d}_k^-} \dots \xrightarrow{\mathfrak{d}_{m-1}^-} \begin{matrix} C(\Gamma_{m,0})\{q\} \\ \oplus \\ C(\Gamma_{m-1,1})\{q\} \end{matrix} \xrightarrow{\mathfrak{d}_m^-} C(\Gamma_{m,1}) \rightarrow 0,$$

where $k_0 = \max\{m-n, 0\}$ as above and

$$\begin{aligned} \mathfrak{d}_{k_0}^- &= \begin{pmatrix} d_{k_0}^- \\ \chi^0 \end{pmatrix}, \\ \mathfrak{d}_k^- &= \begin{pmatrix} d_k^- & 0 \\ \chi^0 & -d_{k-1}^- \end{pmatrix} \text{ for } k_0 < k < m, \\ \mathfrak{d}_m^- &= \begin{pmatrix} \chi^0 & -d_{m-1}^- \end{pmatrix}. \end{aligned}$$

From Decomposition (IV) (more specifically, (12.14)), we have

$$C(\Gamma_{k,0}) \simeq C(\tilde{\Gamma}_k) \oplus C(\Gamma'_k)\{[m-k]\}.$$

By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have

$$C(\Gamma_{k,1}) \simeq C(\Gamma'_k)\{[m+1-k]\} \cong C(\Gamma'_k)\{q^{k-m}\} \oplus C(\Gamma'_k)\{[m-k] \cdot q\}.$$

Therefore,

$$\begin{aligned} & \begin{matrix} C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma_{k,0})\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma_{k-1,1})\{q^{m+1-k}\} \end{matrix} \simeq \begin{matrix} C(\Gamma'_k)\{[m-k]q^{m+1-k}\} \\ \oplus \\ C(\Gamma'_{k-1})\{[m+1-k]q^{m+2-k}\} \end{matrix} \quad \text{for } k_0 < k < m, \end{aligned}$$

and

$$C(\Gamma_{k_0,0})\{q^{m+1-k_0}\} \simeq \frac{C(\tilde{\Gamma}_{k_0})\{q^{m+1-k_0}\} \oplus C(\Gamma'_{k_0})\{[m-k_0]q^{m+1-k_0}\}}{C(\Gamma'_{k_0})\{[m-k_0]q^{m+1-k_0}\}}.$$

So, $\hat{C}(D_{1,1}^-)$ is isomorphic to

$$0 \rightarrow \frac{C(\tilde{\Gamma}_{k_0})\{q^{m+1-k_0}\} \oplus C(\Gamma'_{k_0})\{[m-k_0]q^{m+1-k_0}\}}{C(\Gamma'_{k_0})\{[m-k_0]q^{m+1-k_0}\}} \xrightarrow{\mathfrak{d}_{k_0}^- \dots \mathfrak{d}_{k-1}^-} \frac{C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \oplus C(\Gamma'_k)\{[m-k]q^{m+1-k}\} \oplus C(\Gamma'_{k-1}) \oplus C(\Gamma'_{k-1})\{[m+1-k]q^{m+2-k}\}}{C(\Gamma'_{k-1})\{[m+1-k]q^{m+2-k}\}} \xrightarrow{\mathfrak{d}_k^- \dots \mathfrak{d}_{m-1}^-} \frac{C(\tilde{\Gamma}_m)\{q\} \oplus C(\Gamma'_{m-1}) \oplus C(\Gamma'_{m-1})\{q^2\}}{C(\Gamma'_{m-1})\{q^2\}} \xrightarrow{\mathfrak{d}_m^-} C(\Gamma_{m,1}) \rightarrow 0.$$

In this form, for $k_0 < k < m-1$, \mathfrak{d}_k^- is given by a 4×4 matrix $(\mathfrak{d}_{k;i,j}^-)_{4 \times 4}$. Clearly,

$$\mathfrak{d}_{k;i,j}^- = 0 \text{ for } (i,j) = (1,3), (1,4), (2,3), (2,4).$$

By Lemma 12.11,

$$\mathfrak{d}_{k;1,1}^- \approx \tilde{d}_k^-.$$

By Lemma 12.8,

$$\mathfrak{d}_{k;3,3}^- \approx \delta_{k-1}^-.$$

By (12.19) in Corollary 12.7 and that $\vec{\beta}_k \circ f_k \simeq 0$, we know that

$$\begin{aligned} \mathfrak{d}_{k;4,1}^- &\simeq 0, \\ \mathfrak{d}_{k;4,2}^- &\approx \text{id}_{C(\Gamma'_k)\{[m-k]q^{m+1-k}\}}. \end{aligned}$$

By Lemma 12.20, we have

$$\mathfrak{d}_{k;4,3}^- \simeq 0.$$

Altogether, we have that, for $k_0 < k < m-1$,

$$\mathfrak{d}_k^- \simeq \begin{pmatrix} c_k \tilde{d}_k^- & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & c'_k \delta_{k-1}^- & * \\ 0 & c''_k \text{id}_{C(\Gamma'_k)\{[m-k]q^{m+1-k}\}} & 0 & * \end{pmatrix},$$

where c_k , c'_k and c''_k are non-zero scalars and $*$ means morphisms we have not determined. Similarly,

$$\begin{aligned} \mathfrak{d}_{k_0}^- &\simeq \begin{pmatrix} c_{k_0} \tilde{d}_{k_0}^- & * & & \\ * & * & & \\ * & * & & \\ 0 & c''_{k_0} \text{id}_{C(\Gamma'_{k_0})\{[m-k_0]q^{m+1-k_0}\}} & & \end{pmatrix}, \\ \mathfrak{d}_{m-1}^+ &\simeq \begin{pmatrix} c_{m-1} \tilde{d}_{m-1}^- & * & 0 & 0 \\ * & * & c'_{m-1} \delta_{m-2}^- & * \\ 0 & c''_{m-1} \text{id}_{C(\Gamma'_{m-1})\{q^2\}} & 0 & * \end{pmatrix}, \end{aligned}$$

where c_{k_0} , c''_{k_0} , c_{m-1} , c'_{m-1} and c''_{m-1} are non-zero scalars.

Now apply Gaussian Elimination (Lemma 12.19) to $c''_{k_0} \text{id}_{C(\Gamma'_{k_0})\{[m-k_0]q^{m+1-k_0}\}}$ in \mathfrak{d}_k^- for $k = k_0, k_0+1, \dots, m-1$ in that order. We get that $\hat{C}(D_{11}^-)$ is homotopic to

$$0 \rightarrow C(\tilde{\Gamma}_{k_0})\{q^{m+1-k_0}\} \xrightarrow{\mathfrak{d}_{k_0}^-} \dots \xrightarrow{\mathfrak{d}_{k-1}^-} \frac{C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \oplus C(\Gamma'_{k-1})}{C(\Gamma'_{k-1})} \xrightarrow{\mathfrak{d}_k^-} \dots \xrightarrow{\mathfrak{d}_{m-1}^-} \frac{C(\tilde{\Gamma}_m)\{q\} \oplus C(\Gamma'_{m-1})}{C(\Gamma'_{m-1})} \xrightarrow{\mathfrak{d}_m^-} C(\Gamma_{m,1}) \rightarrow 0,$$

where

$$(12.28) \quad \hat{\mathfrak{d}}_k^- \simeq \begin{pmatrix} c_k \tilde{d}_k^- & 0 \\ * & c'_k \delta_{k-1}^- \end{pmatrix} \text{ for } k_0 < k < m,$$

$$(12.29) \quad \hat{\mathfrak{d}}_{k_0}^- \simeq \begin{pmatrix} c_{k_0} \tilde{d}_{k_0}^- \\ * \\ * \end{pmatrix}.$$

Next we determine $\hat{\mathfrak{d}}_m^-$. By Decomposition (V) (more specifically, (12.20)), we have

$$C(\Gamma_{m,1}) \simeq \begin{matrix} C(\tilde{\Gamma}_{m+1}) \\ \oplus \\ C(\Gamma''_{m-1}) \end{matrix}.$$

Under this decomposition, $\hat{\mathfrak{d}}_m^-$ is represented by a 2×2 matrix. By Lemmas 12.12, 12.17 and 12.18, we know that

$$(12.30) \quad \hat{\mathfrak{d}}_m^- \simeq \begin{pmatrix} c_m \tilde{d}_m^- & 0 \\ * & c'_m P_{m-1, m-1} \end{pmatrix},$$

where c_m and c'_m are non-zero scalars. So $\hat{C}(D_{11}^-)$ is homotopic to

$$0 \rightarrow C(\tilde{\Gamma}_{k_0})\{q^{m+1-k_0}\} \xrightarrow{\hat{\mathfrak{d}}_{k_0}^-} \dots \xrightarrow{\hat{\mathfrak{d}}_{k-1}^-} \begin{matrix} C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma'_{k-1}) \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_k^-} \dots \xrightarrow{\hat{\mathfrak{d}}_{m-1}^-} \begin{matrix} C(\tilde{\Gamma}_m)\{q\} \\ \oplus \\ C(\Gamma'_{m-1}) \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_m^-} \begin{matrix} C(\tilde{\Gamma}_{m+1}) \\ \oplus \\ C(\Gamma''_{m-1}) \end{matrix} \rightarrow 0,$$

where $\hat{\mathfrak{d}}_m^-, \dots, \hat{\mathfrak{d}}_{k_0}^-$ are given in (12.28), (12.29) and (12.30).

Recall that, by Decomposition (V) (more specifically, Lemma 11.19),

$$C(\Gamma'_k) \simeq \begin{cases} C(\Gamma''_k) \oplus C(\Gamma''_{k-1}) & \text{if } k_0 + 1 \leq k \leq m-1, \\ C(\Gamma''_k) & \text{if } k = k_0. \end{cases}$$

By Proposition 11.25, under the decomposition

$$\begin{matrix} C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma'_{k-1}) \end{matrix} \simeq \begin{matrix} C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma''_{k-1}) \\ \oplus \\ C(\Gamma''_{k-2}) \end{matrix},$$

we have

$$(12.31) \quad \hat{\mathfrak{d}}_m^- \simeq \begin{pmatrix} c_m \tilde{d}_m^- & 0 & 0 \\ * & c'''_m \text{id}_{C(\Gamma''_{m-1})} & 0 \\ * & * & 0 \end{pmatrix},$$

$$(12.32) \quad \hat{\mathfrak{d}}_k^- \simeq \begin{pmatrix} c_k \tilde{d}_k^- & 0 & 0 \\ * & 0 & 0 \\ * & c'''_k \text{id}_{C(\Gamma''_{k-1})} & 0 \end{pmatrix} \text{ for } k_0 + 1 < k < m,$$

where c'''_k is a non-zero scalar for $k_0 + 1 < k \leq m$. Since $C(\Gamma'_{k_0}) \simeq C(\Gamma''_{k_0})$, we have

$$\begin{matrix} C(\tilde{\Gamma}_{k_0+1})\{q^{m-k_0}\} \\ \oplus \\ C(\Gamma'_{k_0}) \end{matrix} \simeq \begin{matrix} C(\tilde{\Gamma}_{k_0+1})\{q^{m-k_0}\} \\ \oplus \\ C(\Gamma''_{k_0}) \end{matrix}$$

and

$$(12.33) \quad \hat{\mathfrak{d}}_{k_0+1}^- \simeq \begin{pmatrix} c_{k_0+1} \tilde{d}_{k_0+1}^- & 0 \\ * & 0 \\ * & c_{k_0+1}''' \text{id}_{C(\Gamma_{k_0}'')} \end{pmatrix},$$

$$(12.34) \quad \hat{\mathfrak{d}}_{k_0}^- \simeq \begin{pmatrix} c_{k_0} \tilde{d}_{k_0+1}^+ \\ * \end{pmatrix},$$

where c_{k_0+1}''' is a non-zero scalar. Putting these together, we know that $\hat{C}(D_{11}^+)$ is homotopic to

$$0 \rightarrow C(\tilde{\Gamma}_{k_0})\{q^{m+1-k_0}\} \xrightarrow{\hat{\mathfrak{d}}_{k_0}^-} \begin{matrix} C(\tilde{\Gamma}_{k_0+1})\{q^{m-k_0}\} \\ \oplus \\ C(\Gamma_{k_0}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_{k_0+1}^-} \dots \xrightarrow{\hat{\mathfrak{d}}_{k-1}^-} \begin{matrix} C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \\ \oplus \\ C(\Gamma_{k-1}'') \\ \oplus \\ C(\Gamma_{k-2}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_k^-} \dots \xrightarrow{\hat{\mathfrak{d}}_{m-1}^-} \begin{matrix} C(\tilde{\Gamma}_m)\{q\} \\ \oplus \\ C(\Gamma_{m-1}'') \\ \oplus \\ C(\Gamma_{m-2}'') \end{matrix} \xrightarrow{\hat{\mathfrak{d}}_m^-} \begin{matrix} C(\tilde{\Gamma}_{m+1}) \\ \oplus \\ C(\Gamma_{m-1}'') \end{matrix} \rightarrow 0,$$

where $\hat{\mathfrak{d}}_m^-, \dots, \hat{\mathfrak{d}}_{k_0}^-$ are given in (12.31), (12.32), (12.33) and (12.34).

Applying Gaussian Elimination (Lemma 12.19) to $c_k''' \text{id}_{C(\Gamma_{k-1}'')}$ in $\hat{\mathfrak{d}}_k^-$ for $k = m, m-1, \dots, k_0+1$, we get that $\hat{C}(D_{11}^-)$ is homotopic to

$$0 \rightarrow C(\tilde{\Gamma}_{k_0})\{q^{m+1-k_0}\} \xrightarrow{\tilde{\mathfrak{d}}_{k_0}^-} \dots \xrightarrow{\tilde{\mathfrak{d}}_{k-1}^-} C(\tilde{\Gamma}_k)\{q^{m+1-k}\} \xrightarrow{\tilde{\mathfrak{d}}_k^-} \dots \xrightarrow{\tilde{\mathfrak{d}}_m^-} C(\tilde{\Gamma}_{m+1}) \rightarrow 0,$$

where $\tilde{\mathfrak{d}}_k^- \simeq c_k \tilde{d}_k^-$ for $k = m, \dots, k_0$. Recall that $c_k \neq 0$ for $k = m, \dots, k_0$. So this last chain complex is isomorphic to $\hat{C}(D_{10}^-)$ in $\text{Ch}^b(\text{hmf})$. Therefore, $\hat{C}(D_{11}^-) \simeq \hat{C}(D_{10}^-)$. \square

So we finished proving (12.2), i.e. $\hat{C}(D_{10}^\pm) \simeq \hat{C}(D_{11}^\pm)$ if $l = 1$. The proof of the rest of Proposition 12.2 is very similar and left to the reader. This completes the proof of Theorem 12.1.

13. INVARIANCE UNDER REIDEMEISTER MOVES

In this section, we prove that the homotopy type of the chain complex associated to a knotted MOY graph is invariant under Reidemeister moves. The main result of this section is Theorem 13.1 below. Note that Theorem 1.1 is a spacial case of Theorem 13.1.

Theorem 13.1. *Let D_0 and D_1 be two knotted MOY graphs. Assume that there is a finite sequence of Reidemeister moves that changes D_0 into D_1 . Then $C(D_0) \simeq C(D_1)$, that is, they are isomorphic as objects of $\text{hCh}^b(\text{hmf})$.*

Theorem 13.1 follows from Lemmas 13.4 and 13.8 below, in which we establish the invariance of the homotopy type under Reidemeister moves I, II_a, II_b and III given in Figures 104, 105, 106 and 107. The proofs of these lemmas are based on an induction on the (highest) color of the edges involved in the Reidemeister move. The starting point of our induction is the following theorem by Khovanov and Rozansky [18].

Theorem 13.2. [18, Theorem 2] *Let D_0 and D_1 be two knotted MOY graphs. Assume that there is a Reidemeister move changing D_0 into D_1 that involves only edges colored by 1. Then $C(D_0) \simeq C(D_1)$, that is, they are isomorphic as objects of $\text{hCh}^b(\text{hmf})$.*

Remark 13.3. The original statement of [18, Theorem 2] covers only link diagrams colored entirely by 1. (c.f. Corollary 11.28.) But its proof in [18, Section 8] is local in the sense that it is based on simplification of the chain complex associated to the part of the link diagram involved in the Reidemeister move. So the slightly more general statement of Theorem 13.2 also follows from this proof.

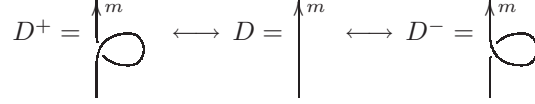


FIGURE 104. Reidemeister Move I

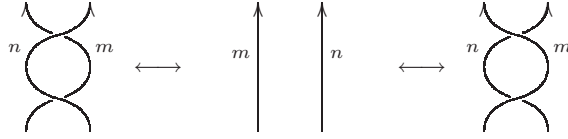
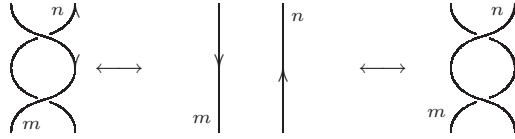
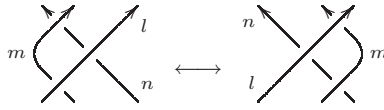
FIGURE 105. Reidemeister Move II_aFIGURE 106. Reidemeister Move II_b

FIGURE 107. Reidemeister Move III

13.1. Invariance under Reidemeister moves II_a, II_b and III. With the invariance under fork sliding (Theorem 12.1) in hand, we can easily prove the invariance of the homotopy type under Reidemeister moves II_a, II_b, III by an induction using the “sliding bi-gon” method introduced in [29] (and used in [27].) The proof of invariance under Reidemeister move I is somewhat different and is postponed to the next subsection.

Lemma 13.4. *Let D_0 and D_1 be two knotted MOY graphs. Assume that there is a Reidemeister move of type II_a, II_b or III that changes D_0 into D_1 . Then $C(D_0) \simeq C(D_1)$, that is, they are isomorphic as objects of $\mathbf{hCh}^b(\mathbf{hmf})$.*



FIGURE 108.

Proof. The proofs for Reidemeister moves II_a , II_b and III are quite similar. We only give details for Reidemeister move II_a here and leave the other two moves to the reader.

Let D_0 and D_1 be the knotted MOY graphs in Figure 108. We prove by induction on k that $C(D_0) \simeq C(D_1)$ if $1 \leq m, n \leq k$. If $k = 1$, then this statement is true because it is a special case of Theorem 13.2. Assume that this statement is true for some $k \geq 1$.

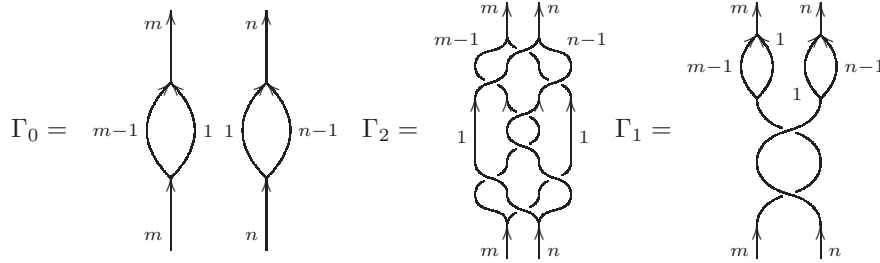


FIGURE 109.

Now consider $k + 1$. Assume that $1 \leq m, n \leq k + 1$ in D_0 and D_1 . Let Γ_0 , Γ_1 and Γ_2 be in the knotted MOY graphs in Figure 109. Here, in case m or $n = 1$, we use the convention that an edge colored by 0 is an edge that does not exist. By Decomposition (II) (Theorem 5.14), we know that $\hat{C}(\Gamma_0) \simeq \hat{C}(D_0)\{[m][n]\}$ and $\hat{C}(\Gamma_1) \simeq \hat{C}(D_1)\{[m][n]\}$. Note that $m - 1, n - 1 \leq k$. By induction hypothesis and the normalization in Definition 11.16, we know that $\hat{C}(\Gamma_0) \simeq \hat{C}(\Gamma_2)$. By the invariance under fork sliding (Theorem 12.1), we know that $\hat{C}(\Gamma_1) \simeq \hat{C}(\Gamma_2)$. Thus, $\hat{C}(\Gamma_0) \simeq \hat{C}(\Gamma_1)$. By Proposition 3.20, it follows that $\hat{C}(D_0) \simeq \hat{C}(D_1)$, which, by the normalization in Definition 11.16, is equivalent to $C(D_0) \simeq C(D_1)$. This completes the induction. \square

13.2. Invariance under Reidemeister move I. The proof of invariance under Reidemeister move I is somewhat different from that under Reidemeister moves II and III. The basic idea is still the “sliding bi-gon”. But we also need to do some “untwisting” to get the invariance.

Lemma 13.5. *Let $\Gamma_{m,n}$ be the MOY graph in Figure 110. Then*

$$\begin{aligned} & \text{Hom}_{\text{HMF}}(C(\Gamma_{m,n}), C(\Gamma_{m,n})) \cong \text{Hom}_{\text{HMF}}(C(\bar{\Gamma}_{m,n}), C(\bar{\Gamma}_{m,n})) \\ & \cong C(\emptyset) \left\{ \begin{bmatrix} N \\ m+n \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix} q^{(m+n)(N-m-n)+mn} \right\}, \end{aligned}$$

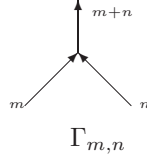


FIGURE 110.

where $\bar{\Gamma}_{m,n}$ is $\Gamma_{m,n}$ with reversed orientation. In particular, the lowest non-vanishing quantum degree of these spaces is 0. Therefore, for $k < l$,

$$\text{Hom}_{\text{hmf}}(C(\Gamma_{m,n})\{q^k\}, C(\Gamma_{m,n})\{q^l\}) \cong \text{Hom}_{\text{hmf}}(C(\bar{\Gamma}_{m,n})\{q^k\}, C(\bar{\Gamma}_{m,n})\{q^l\}) \cong 0.$$

Proof. Consider the MOY graph Γ in Figure 28. It is easy to check that

$$\begin{aligned} \text{Hom}_{\text{HMF}}(C(\Gamma_{m,n}), C(\Gamma_{m,n})) &\cong \text{Hom}_{\text{HMF}}(C(\bar{\Gamma}_{m,n}), C(\bar{\Gamma}_{m,n})) \\ &\cong H(\Gamma) \langle m+n \rangle \{q^{(m+n)(N-m-n)+mn}\}. \end{aligned}$$

By Lemma 8.3,

$$H(\Gamma) \cong C(\emptyset) \langle m+n \rangle \left\{ \begin{bmatrix} N \\ m+n \end{bmatrix} \begin{bmatrix} m+n \\ n \end{bmatrix} \right\},$$

whose lowest non-vanishing quantum grading is $-(m+n)(N-m-n)-mn$. This implies the lemma. \square

The next lemma is [44, Proposition 6.1]. The special case when $m = n = 1$ of this lemma has appeared in [33]. For the convenience of the reader, we prove this lemma here instead of just citing [44].

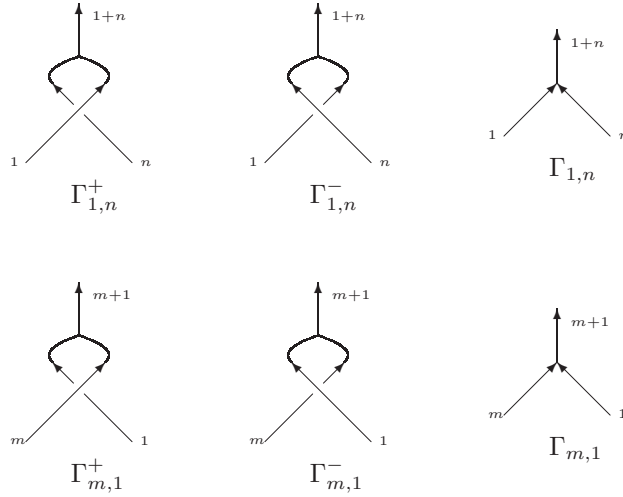


FIGURE 111.

Lemma 13.6. [44, Proposition 6.1] Let $\Gamma_{1,n}^\pm$ and $\Gamma_{m,1}^\pm$ be the twisted MOY graphs in Figure 111. Then

$$\begin{aligned} \hat{C}(\Gamma_{1,n}^+) &\simeq \hat{C}(\Gamma_{1,n})\{q^n\}, & \hat{C}(\Gamma_{1,n}^-) &\simeq \hat{C}(\Gamma_{1,n})\{q^{-n}\}, \\ \hat{C}(\Gamma_{m,1}^+) &\simeq \hat{C}(\Gamma_{m,1})\{q^m\}, & \hat{C}(\Gamma_{m,1}^-) &\simeq \hat{C}(\Gamma_{m,1})\{q^{-m}\}, \end{aligned}$$

where “ \simeq ” is the isomorphism in $\mathbf{hCh}^b(\mathbf{hmf})$.

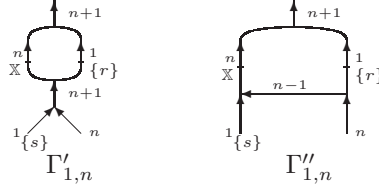


FIGURE 112.

Proof. We only give details for $\hat{C}(\Gamma_{1,n}^\pm) \simeq \hat{C}(\Gamma_{1,n})\{q^{\pm n}\}$ here. The proof for $\hat{C}(\Gamma_{m,1}^\pm) \simeq \hat{C}(\Gamma_{1,n})\{q^{\pm m}\}$ is very similar and left to the reader. Recall that

$$\hat{C}(\Gamma_{1,n}) = “0 \rightarrow C(\Gamma_{1,n}) \rightarrow 0.”$$

Let $\Gamma'_{1,n}$ and $\Gamma''_{1,n}$ be the MOY graphs in Figure 112. Then, by Corollary 11.28,

$$\begin{aligned} \hat{C}(\Gamma_{1,n}^+) &= “0 \rightarrow C(\Gamma'_{1,n}) \xrightarrow{\chi^1} C(\Gamma''_{1,n})\{q^{-1}\} \rightarrow 0”, \\ \hat{C}(\Gamma_{1,n}^-) &= “0 \rightarrow C(\Gamma''_{1,n})\{q\} \xrightarrow{\chi^0} C(\Gamma'_{1,n}) \rightarrow 0”, \end{aligned}$$

where χ^0 and χ^1 are induced by the apparent changes of MOY graph.

Note that $\Gamma'_{1,n}$ is obtained from $\Gamma_{1,n}$ by an edge splitting. Denote by $C(\Gamma_{1,n}) \xrightarrow{\phi} C(\Gamma'_{1,n})$ and $C(\Gamma'_{1,n}) \xrightarrow{\bar{\phi}} C(\Gamma_{1,n})$ the morphisms induced by this edge splitting and its reverse edge merging. By Decomposition (II) (Theorem 5.14), we know that

$$(13.1) \quad C(\Gamma'_{1,n}) \simeq C(\Gamma_{1,n})\{[n+1]\} = \bigoplus_{j=0}^n C(\Gamma_{1,n})\{q^{-n+2j}\}.$$

It is not hard to explicitly write down inclusions and projections in this decomposition. For $j = 0, \dots, n$, define $\alpha_j = \mathbf{m}(r^j) \circ \phi$ and $\beta_j = \bar{\phi} \circ \mathbf{m}(X_{n-j})$, where X_k is the k -th elementary symmetric polynomial in \mathbb{X} . Then $C(\Gamma_{1,n})\{q^{-n+2j}\} \xrightarrow{\alpha_j} C(\Gamma'_{1,n})$ and $C(\Gamma'_{1,n}) \xrightarrow{\beta_j} C(\Gamma_{1,n})\{q^{-n+2j}\}$ are homogeneous morphisms preserving both gradings. And, by Lemma 7.11,

$$\beta_j \circ \alpha_i \approx \begin{cases} \text{id}_{C(\Gamma_{1,n})\{q^{-n+2j}\}} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, α_i and β_j give the inclusions and projections in the decomposition (13.1).

By Corollary 5.13 and Decomposition (II) (Theorem 5.14), we have

$$(13.2) \quad C(\Gamma''_{1,n}) \simeq C(\Gamma_{1,n})\{[n]\} = \bigoplus_{j=0}^{n-1} C(\Gamma_{1,n})\{q^{-n+1+2j}\}.$$

By decompositions (13.1) and (13.2), $\hat{C}(\Gamma_{1,n}^+)$ is isomorphic to

$$0 \rightarrow \begin{array}{c} C(\Gamma_{1,n})\{q^{-n}\} \\ \oplus \\ C(\Gamma_{1,n})\{q^{-n+2}\} \\ \oplus \\ \vdots \\ \oplus \\ C(\Gamma_{1,n})\{q^n\} \end{array} \xrightarrow{\chi^1} \begin{array}{c} C(\Gamma_{1,n})\{q^{-n}\} \\ \oplus \\ C(\Gamma_{1,n})\{q^{-n+2}\} \\ \oplus \\ \vdots \\ \oplus \\ C(\Gamma_{1,n})\{q^{n-2}\} \end{array} \rightarrow 0,$$

where χ^1 is represented by a $n \times (n+1)$ matrix $(\chi_{i,j}^1)_{n \times (n+1)}$. By Lemma 13.5, we have that

$$(13.3) \quad \chi_{i,j}^1 \simeq 0 \text{ if } i > j.$$

Similarly, $\hat{C}(\Gamma_{1,n}^-)$ is isomorphic to

$$0 \rightarrow \begin{array}{c} C(\Gamma_{1,n})\{q^{-n+2}\} \\ \oplus \\ C(\Gamma_{1,n})\{q^{-n+4}\} \\ \oplus \\ \vdots \\ \oplus \\ C(\Gamma_{1,n})\{q^n\} \end{array} \xrightarrow{\chi^0} \begin{array}{c} C(\Gamma_{1,n})\{q^{-n}\} \\ \oplus \\ C(\Gamma_{1,n})\{q^{-n+2}\} \\ \oplus \\ \vdots \\ \oplus \\ C(\Gamma_{1,n})\{q^n\} \end{array} \rightarrow 0,$$

where χ^0 is represented by a $(n+1) \times n$ matrix $(\chi_{i,j}^0)_{(n+1) \times n}$. By Lemma 13.5, we have that

$$(13.4) \quad \chi_{i,j}^0 \simeq 0 \text{ if } i > j+1.$$

Consider the composition $\beta_{j+1} \circ \chi^0 \circ \chi^1 \circ \alpha_j$. On the one hand, by Lemma 7.11 and Corollary 8.9, we have

$$\beta_{j+1} \circ \chi^0 \circ \chi^1 \circ \alpha_j \approx \beta_{j+1} \circ \mathbf{m}(r-s) \circ \alpha_j \approx \bar{\phi} \circ \mathbf{m}(X_{n-j-1} r^j (r-s)) \circ \phi \approx \text{id}_{C(\Gamma_{1,n})}.$$

On the other hand, by (13.3) and (13.4), we have

$$\beta_{j+1} \circ \chi^0 \circ \chi^1 \circ \alpha_j \approx \sum_{k=1}^{n-1} \chi_{j+1,k}^0 \circ \chi_{k,j}^1 \simeq \chi_{j+1,j}^0 \circ \chi_{j,j}^1.$$

So, $\chi_{j+1,j}^0 \circ \chi_{j,j}^1 \approx \text{id}_{C(\Gamma_{1,n})}$. This shows that $\chi_{j+1,j}^0$ and $\chi_{j,j}^1$ are both isomorphisms in hmf.

Using (13.3), we apply Gaussian Elimination (Lemma 12.19) to $\chi_{j,j}^1$ in $\hat{C}(\Gamma_{1,n}^+)$ for $j = 1, 2, \dots, n$ in that order. This reduces $\hat{C}(\Gamma_{1,n}^+)$ to

$$0 \rightarrow C(\Gamma_{1,n})\{q^n\} \rightarrow 0.$$

So $\hat{C}(\Gamma_{1,n}^+) \simeq \hat{C}(\Gamma_{1,n})\{q^n\}$. Similarly, using (13.4), we apply Gaussian Elimination (Lemma 12.19) to $\chi_{j+1,j}^0$ in $\hat{C}(\Gamma_{1,n}^-)$ for $j = n, n-1, \dots, 1$ in that order. This reduces $\hat{C}(\Gamma_{1,n}^-)$ to

$$0 \rightarrow C(\Gamma_{1,n})\{q^{-n}\} \rightarrow 0.$$

So $\hat{C}(\Gamma_{1,n}^-) \simeq \hat{C}(\Gamma_{1,n})\{q^{-n}\}$. □

Lemma 13.7. *Let D^+ , D^- and D be the knotted MOY graphs in Figure 104. Then*

$$(13.5) \quad \hat{C}(D^+) \simeq \hat{C}(D) \langle m \rangle \| m \| \{q^{-m(N+1-m)}\},$$

$$(13.6) \quad \hat{C}(D^-) \simeq \hat{C}(D) \langle m \rangle \| -m \| \{q^{m(N+1-m)}\},$$

where $\| * \|$ means shifting the homological grading.

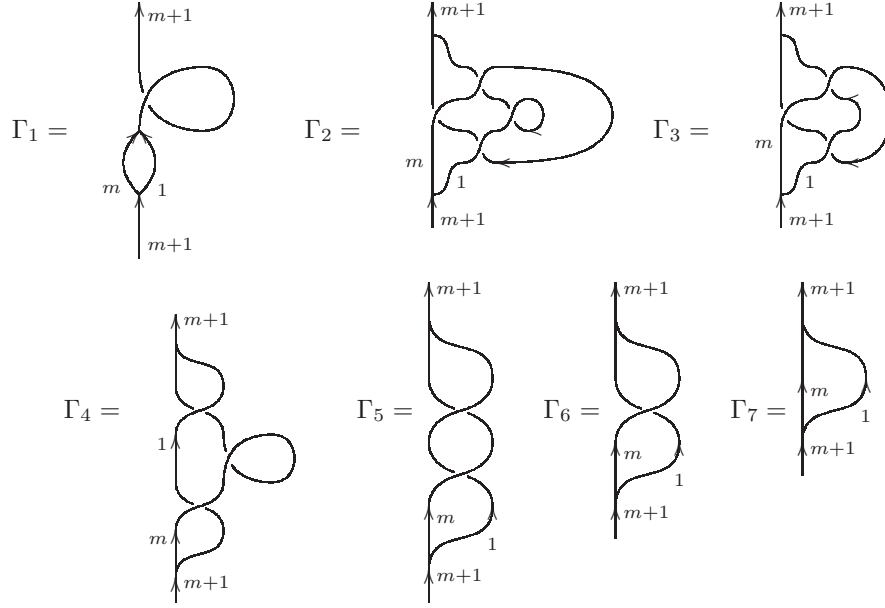


FIGURE 113.

Proof. We prove (13.5) by an induction on m . The proof of (13.6) is similar and left to the reader.

If $m = 1$, then (13.5) follows from [18, Theorem 2]. (See Theorem 13.2 above.) Assume that (13.5) is true for some $m \geq 1$. Let us prove (13.5) for $m + 1$.

Consider the knotted MOY graphs $\Gamma_1, \dots, \Gamma_7$ in Figure 113. By Decomposition (II) (Theorem 5.14), we have

$$\hat{C}(\Gamma_1) \simeq \hat{C}(D^+) \{[m+1]\} \quad \text{and} \quad \hat{C}(\Gamma_7) \simeq \hat{C}(D) \{[m+1]\}.$$

By Theorem 12.1, we have $\hat{C}(\Gamma_1) \simeq \hat{C}(\Gamma_2)$. Since (13.5) is true for 1, we know that $\hat{C}(\Gamma_2) \simeq \hat{C}(\Gamma_3) \langle 1 \rangle \| 1 \| \{q^{-N}\}$. From Lemma 13.4, one can see that $\hat{C}(\Gamma_3) \simeq \hat{C}(\Gamma_4)$. Since (13.5) is true for m , we know that $\hat{C}(\Gamma_4) \simeq \hat{C}(\Gamma_5) \langle m \rangle \| m \| \{q^{-m(N+1-m)}\}$. By Lemma 13.6, we know that $\hat{C}(\Gamma_5) \simeq \hat{C}(\Gamma_6) \{q^m\}$ and $\hat{C}(\Gamma_6) \simeq \hat{C}(\Gamma_7) \{q^m\}$. Putting these together, we get that

$$\hat{C}(\Gamma_1) \simeq \hat{C}(\Gamma_7) \langle m+1 \rangle \| m+1 \| \{q^{-(m+1)(N-m)}\}.$$

From Proposition 3.20, it follows that (13.5) is true for $m + 1$. This completes the induction. \square

Lemma 13.8. *Let D_0 and D_1 be two knotted MOY graphs. Assume that there is a Reidemeister move of type I that changes D_0 into D_1 . Then $C(D_0) \simeq C(D_1)$, that is, they are isomorphic as objects of $\mathbf{hCh}^b(\mathbf{hmf})$.*

Proof. The lemma follows easily from Lemma 13.7 and the normalization in Definition 11.16. \square

14. THE EULER CHARACTERISTIC AND THE \mathbb{Z}_2 -GRADING

In this section, we determine the Euler characteristic and the \mathbb{Z}_2 -grading of the colored $\mathfrak{sl}(N)$ -homology and prove Theorem 1.3. We start by reviewing the alternative construction of the Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial for links colored by positive integers given in [29].

14.1. The MOY construction of the Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial.

In [29], Murakami, Ohtsuki and Yamada gave an alternative construction of the Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial for links colored by positive integers. We review their construction in this subsection. The notations and normalizations we use here are slightly different from that used in [29].

Define $\mathcal{N} = \{-N+1, -N+3, \dots, N-3, N-1\}$ and $\mathcal{P}(\mathcal{N})$ to be the set of subsets of \mathcal{N} . For a finite set A , denote by $\#A$ the number of elements of A . Define a function $\pi : \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\pi(A_1, A_2) = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} \text{ for } A_1, A_2 \in \mathcal{P}(\mathcal{N}).$$

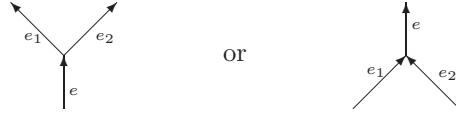


FIGURE 114.

Let Γ be a closed trivalent MOY graph, and $E(\Gamma)$ the set of edges of Γ . Denote by $c : E(\Gamma) \rightarrow \mathbb{N}$ the color function of Γ . That is, for every edge e of Γ , $c(e) \in \mathbb{N}$ is the color of e . A state of Γ is a function $\sigma : E(\Gamma) \rightarrow \mathcal{P}(\mathcal{N})$ such that

- (i) for every edge e of Γ , $\#\sigma(e) = c(e)$,
- (ii) for every vertex v of Γ , as depicted in Figure 114, we have $\sigma(e) = \sigma(e_1) \cup \sigma(e_2)$.
(In particular, this implies that $\sigma(e_1) \cap \sigma(e_2) = \emptyset$.)

For a state σ of Γ and a vertex v of Γ (as depicted in Figure 114), the weight of v with respect to σ is defined to be

$$\text{wt}(v; \sigma) = q^{\frac{c(e_1)c(e_2)}{2} - \pi(\sigma(e_1), \sigma(e_2))}.$$

Given a state σ of Γ , replace each edge e of Γ by $c(e)$ parallel edges, assign to each of these new edges a different element of $\sigma(e)$ and, at every vertex, connect each pair of new edges assigned the same element of \mathcal{N} . This changes Γ into a collection \mathcal{C} of embedded circles, each of which is assigned an element of \mathcal{N} . By abusing notation, we denote by $\sigma(C)$ the element of \mathcal{N} assigned to $C \in \mathcal{C}$. Note that:

- There may be intersections between different circles in \mathcal{C} . But, each circle in \mathcal{C} is embedded, that is, it has no self-intersections or self-tangencies.

$$\begin{aligned}
(6) \quad & \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_N = \begin{bmatrix} m-1 \\ n \end{bmatrix} \cdot \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_N + \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \cdot \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right\rangle_N. \\
(7) \quad & \left\langle \begin{array}{c} \text{Diagram 4} \end{array} \right\rangle_N = \sum_{j=\max\{m-n, 0\}}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \cdot \left\langle \begin{array}{c} \text{Diagram 5} \end{array} \right\rangle_N.
\end{aligned}$$

(8) The above equations remain true if we reverse the orientation of the whole MOY graph.

Remark 14.3. Although part (6) of Theorem 14.2, which corresponds to Decomposition (IV), is not explicitly given in [29], it is the $n = 1$ special case of part (7), which is given in [29, Proposition A.10].

Definition 14.4. [29] For a link diagram D colored by positive integers, define $\langle D \rangle_N$ by applying the following at every crossing of D .

$$\begin{aligned}
& \left\langle \begin{array}{c} \text{Crossing 1} \end{array} \right\rangle_N = \sum_{k=\max\{0, m-n\}}^m (-1)^{m-k} q^{k-m} \left\langle \begin{array}{c} \text{Diagram 6} \end{array} \right\rangle_N \\
& \left\langle \begin{array}{c} \text{Crossing 2} \end{array} \right\rangle_N = \sum_{k=\max\{0, m-n\}}^m (-1)^{k-m} q^{m-k} \left\langle \begin{array}{c} \text{Diagram 7} \end{array} \right\rangle_N
\end{aligned}$$

Also, for each crossing c of D , define the shifting factor $s(c)$ of c by

$$s \left(\begin{array}{c} \text{Crossing} \end{array} \right) = \begin{cases} (-1)^{-m} q^{m(N+1-m)} & \text{if } m = n, \\ 1 & \text{if } m \neq n, \end{cases}$$

$$s \left(\begin{array}{c} \text{diagram of a crossing with strands labeled } m \text{ and } n \end{array} \right) = \begin{cases} (-1)^m q^{-m(N+1-m)} & \text{if } m = n, \\ 1 & \text{if } m \neq n. \end{cases}$$

The re-normalized Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial $\text{RT}_D(q)$ of D is defined to be

$$\text{RT}_D(q) = \langle D \rangle_N \cdot \prod_c s(c),$$

where c runs through all crossings of D .

Theorem 14.5. [29] $\langle D \rangle_N$ is invariant under regular Reidemeister moves. $\text{RT}_D(q)$ is invariant under all Reidemeister moves and is a re-normalization of the Reshetikhin-Turaev $\mathfrak{sl}(N)$ -polynomial for links colored by positive integers.

14.2. Proof of Theorem 1.3. Now we are ready to prove Theorem 1.3. First, we introduce the colored rotation number of a closed trivalent MOY graph.

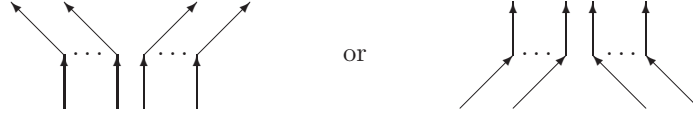


FIGURE 115.

Let Γ be a closed trivalent MOY graph. Replace each edge of Γ of color m by m parallel edges colored by 1 and replace each vertex of Γ , as depicted in Figure 114, to the corresponding configuration in Figure 115, where each strand is an edge colored by 1. This changes Γ into a collection of disjoint embedded circles in the plane.

Definition 14.6. The colored rotation number $\text{cr}(\Gamma)$ of Γ is defined to be the sum of the rotation numbers of these circles.

Recall that the homology $H(\Gamma)$ of a MOY graph Γ is defined in Definition 5.7, and the graded dimension $\text{gdim}(C(\Gamma))$ is defined to be

$$\text{gdim}(C(\Gamma)) = \sum_{\varepsilon, i} \tau^\varepsilon q^i H^{\varepsilon, i}(\Gamma) \in \mathbb{C}[\tau, q]/(\tau^2),$$

where $H^{\varepsilon, i}(\Gamma)$ is the subspace of $H(\Gamma)$ of homogeneous elements of \mathbb{Z}_2 -degree ε and quantum degree i .

The following theorem implies Theorem 1.3.

Theorem 14.7. Let Γ be a closed trivalent MOY graph. Then

- (1) $\text{gdim}(C(\Gamma))|_{\tau=1} = \langle \Gamma \rangle_N$,
- (2) $H^{\varepsilon, i}(\Gamma) = 0$ if $\varepsilon - \text{cr}(\Gamma) = 1$.

Proof. We prove the theorem by a double induction on the highest color of edges of Γ and the number of edges of Γ with the highest color. Consider the following two statements.

- $A_{m,k}$: Part (1) of Theorem 14.7 is true if the color of every edge of Γ is less than or equal to m and Γ has exactly k edges of color m .
- $B_{m,k}$: Part (2) of Theorem 14.7 is true if the color of every edge of Γ is less than or equal to m and Γ has exactly k edges of color m .

Kauffman and Vogel [16] proved that, for closed trivalent MOY graphs colored by 1, 2, the polynomial satisfying the relations in Theorem 14.2 is unique. Since both $\text{gdim}(C(\Gamma))|_{\tau=1}$ and $\langle \Gamma \rangle_N$ satisfy these relations, this implies that the statement $A_{m,k}$ is true for $m = 1, 2$ and all $k \geq 0$. Khovanov and Rozansky [18] proved that $B_{m,k}$ is true for $m = 1, 2$ and all $k \geq 0$. We assume that there is a $m \geq 2$ such that $A_{1,k}, \dots, A_{m,k}, B_{1,k}, \dots, B_{m,k}$ are true for all $k \geq 0$.

Consider the statements $A_{m+1,k}$ and $B_{m+1,k}$. Clearly, by the above assumption, $A_{m+1,0}$ and $B_{m+1,0}$ are true. Assume that there is a $k \geq 0$ such that $A_{m+1,k}$ and $B_{m+1,k}$ are true. Consider $A_{m+1,k+1}$ and $B_{m+1,k+1}$. Assume that the highest color of edges of Γ is $m+1$ and there are exactly $k+1$ edges of color $m+1$ in Γ . Note that Γ is a closed trivalent MOY graph. So an edge of color $m+1$ in Γ is either a circle colored by $m+1$ or has a neighborhood of the type in Figure 116.

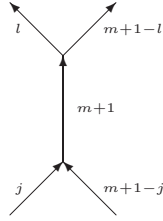


FIGURE 116.

Case 1. Assume that there is a circle \bigcirc_{m+1} colored by $m+1$ in Γ . Let $\tilde{\Gamma}$ be Γ with \bigcirc_{m+1} removed. Then, by $A_{m+1,k}$ and $B_{m+1,k}$, the theorem is true for $\tilde{\Gamma}$. Note that $H(\Gamma) = H(\bigcirc_{m+1}) \otimes_{\mathbb{C}} H(\tilde{\Gamma})$, $\text{cr}(\Gamma) = m+1 + \text{cr}(\tilde{\Gamma})$, $H(\bigcirc_{m+1}) \cong C(\emptyset) \begin{bmatrix} N \\ m+1 \end{bmatrix} \langle m+1 \rangle$ and, by part (1) of Theorem 14.2, $\langle \Gamma \rangle_N = \begin{bmatrix} N \\ m+1 \end{bmatrix} \langle \tilde{\Gamma} \rangle_N$. So

$$\text{gdim}(C(\Gamma))|_{\tau=1} = \begin{bmatrix} N \\ m+1 \end{bmatrix} \text{gdim}(C(\tilde{\Gamma}))|_{\tau=1} = \begin{bmatrix} N \\ m+1 \end{bmatrix} \langle \tilde{\Gamma} \rangle_N = \langle \Gamma \rangle_N,$$

and $H^{\varepsilon,i}(\Gamma) = 0$ if $\varepsilon - m - 1 - \text{cr}(\tilde{\Gamma}) = 1$, that is, $\varepsilon - \text{cr}(\Gamma) = 1$. So the theorem is true for Γ .

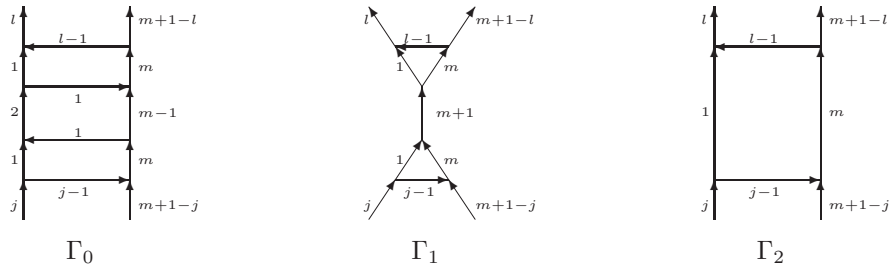


FIGURE 117.

Case 2. Assume there are no circles colored by $m + 1$ in Γ . Then every edge in Γ colored by $m + 1$ is of the form in Figure 116. Let e be such an edges of Γ as depicted in Figure 116. We modify Γ locally near e as depicted in Figure 117. This gives us new MOY graphs Γ_0 , Γ_1 and Γ_2 , which are identical to Γ except in the neighborhoods shown in Figure 117. Note that $\text{cr}(\Gamma_0) = \text{cr}(\Gamma_1) = \text{cr}(\Gamma_2) = \text{cr}(\Gamma)$. Also, each of Γ_0 and Γ_2 has exactly k edges colored by $m + 1$ and, therefore, the theorem is true for Γ_0 and Γ_2 .

By the Bouquet move (Corollary 5.13) and Decomposition (II) (Theorem 5.14), we have

$$(14.2) \quad H(\Gamma_1) \cong H(\Gamma)\{[j] \cdot [l]\}$$

and, by Decomposition (IV) (Theorem 9.1), we have

$$(14.3) \quad H(\Gamma_0) \cong H(\Gamma_1) \oplus H(\Gamma_2)\{[m - 1]\}.$$

Similarly, using parts (2), (4) and (6) of Theorem 14.2, we get

$$(14.4) \quad \langle \Gamma_1 \rangle_N = [j] \cdot [l] \cdot \langle \Gamma \rangle_N,$$

$$(14.5) \quad \langle \Gamma_0 \rangle_N = \langle \Gamma_1 \rangle_N + [m - 1] \cdot \langle \Gamma_2 \rangle_N.$$

From (14.2) and (14.3), we can see that $H^{\varepsilon, i}(\Gamma) \cong 0$ if $\varepsilon - \text{cr}(\Gamma_0) = 1$, that is, if $\varepsilon - \text{cr}(\Gamma) = 1$. Comparing (14.2) and (14.3) to (14.4) and (14.5), we get that

$$\begin{aligned} \text{gdim}(C(\Gamma))|_{\tau=1} &= \frac{\text{gdim}(C(\Gamma_1))|_{\tau=1}}{[j] \cdot [l]} \\ &= \frac{\text{gdim}(C(\Gamma_0))|_{\tau=1} - [m - 1] \cdot \text{gdim}(C(\Gamma_2))|_{\tau=1}}{[j] \cdot [l]} \\ &= \frac{\langle \Gamma_0 \rangle_N - [m - 1] \cdot \langle \Gamma_2 \rangle_N}{[j] \cdot [l]} \\ &= \frac{\langle \Gamma_1 \rangle_N}{[j] \cdot [l]} \\ &= \langle \Gamma \rangle_N \end{aligned}$$

Thus, the theorem is true for Γ .

Combining *Case 1* and *Case 2*, we know that statements $A_{m+1, k+1}$ and $B_{m+1, k+1}$ are true. So $A_{m+1, k}$ and $B_{m+1, k}$ are true for all $k \geq 0$. This completes the induction. So the statements $A_{m, k}$ and $B_{m, k}$ are true for all $m \geq 1$ and $k \geq 0$. \square

Corollary 14.8. *The relations in Theorem 14.2 uniquely determine the $\mathfrak{sl}(N)$ -polynomial $\langle * \rangle_N$ for close trivalent MOY graphs.*

Proof. This corollary can be proved using the inductive argument in the proof of Theorem 14.7. We leave the details to the reader. \square

Proof of Theorem 1.3. First, by comparing Definitions 11.16 and 14.4, we can see that the equation $P_L(1, q, -1) = \text{RT}_L(q)$ follows easily from part (1) of Theorem 14.7.

Next, we consider the \mathbb{Z}_2 -grading. Let D be a diagram of L and Γ be any complete resolution of D . Note that the number $\text{cr}(\Gamma)$ does not depend on the choice of Γ . We define $\text{cr}(D) = \text{cr}(\Gamma)$. At each crossing c of D , define an adjustment

term $\mathbf{a}(c)$ by

$$\mathbf{a} \left(\begin{array}{c} \text{diagram of a crossing with strands labeled } m \text{ and } n \end{array} \right) = \mathbf{a} \left(\begin{array}{c} \text{diagram of a crossing with strands labeled } m \text{ and } n \end{array} \right) = \begin{cases} m & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

Define $\hat{\text{tc}}(D) = \text{cr}(D) + \sum_c \mathbf{a}(c)$, where c runs through all crossings of D . Then, by Definition 11.16 and part (2) of Theorem 14.7,

$$H^{\varepsilon, i, j}(L) = 0 \text{ if } \varepsilon - \hat{\text{tc}}(D) = 1 \in \mathbb{Z}_2,$$

Note that the parity of $\hat{\text{tc}}(D)$ is invariant under Reidemeister moves and unknotting (that is, switching the above- and below- strands at a crossing.) Using these moves, we can change D into a link diagram U without crossings. That is, a collection of disjoint colored circles. It is clear that $\hat{\text{tc}}(U) = \text{tc}(L)$. So, as elements of \mathbb{Z}_2 , $\hat{\text{tc}}(D) = \hat{\text{tc}}(U) = \text{tc}(L)$. This completes the proof. \square

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